Dull Measurements
(aka Weak Measurements)

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This article is a very short pedagogical introduction to the maths of what is commonly called quantum mechanical "weak measurements" (or " weak values"). The article was written as an appendix to a post entitled "Dull Measurements (aka Weak Measurements)" for my blog "Quantum Bayesian Networks" (www.qbnets.wordpress.com)

If A is any operator acting on a Hilbert space $\mathcal{H}$, and $|\psi_\alpha\rangle, |\psi_\beta\rangle \in \mathcal{H}$, then we will use the following notation:

$$\langle A\rangle_{\alpha,\beta} = \langle\psi_\alpha|A|\psi_\beta\rangle \;, \tag{1}$$

$$\langle A\rangle^W_{\alpha,\beta} = \frac{\langle\psi_\alpha|A|\psi_\beta\rangle}{\langle\psi_\alpha|\psi_\beta\rangle} \;, \tag{2}$$

$$\langle 1\rangle_{\alpha,\beta} = \langle\psi_\alpha|\psi_\beta\rangle \;. \tag{3}$$

Let g be a non-negative number, called the coupling constant. It will be our expansion parameter. *Weak Measurement formalism is only valid to first order in g.* That's why it's called weak.

Let $\mathcal{H}_S$ denote the Hilbert space for a system S and $\mathcal{H}_M$ the Hilbert space for a meter M. Suppose P_M is a Hermitian operator acting on $\mathcal{H}_M$, and A_S is a Hermitian operator acting on $\mathcal{H}_S$. Suppose P_M and A_S commute. Suppose that $|\psi^S_{\sigma_j}\rangle \in \mathcal{H}_S$ and $|\psi^M_{\mu_j}\rangle \in \mathcal{H}_M$ for $j = 1, 2$. Then, to first order in g, one has

$$\begin{array}{l}\langle\psi^S_{\sigma_2}|\\ \langle\psi^M_{\mu_2}|\end{array} e^{igP_M A_S} \begin{array}{l}|\psi^S_{\sigma_1}\rangle\\ |\psi^M_{\mu_1}\rangle\end{array} \approx \langle 1_M\rangle_{\mu_2,\mu_1}\langle 1_S\rangle_{\sigma_2,\sigma_1} + ig\langle P_M\rangle_{\mu_2,\mu_1}\langle A_S\rangle_{\sigma_2,\sigma_1} \tag{4a}$$

$$= \langle 1_M\rangle_{\mu_2,\mu_1}\langle 1_S\rangle_{\sigma_2,\sigma_1}\left[1 + ig\langle P_M\rangle^W_{\mu_2,\mu_1}\langle A_S\rangle^W_{\sigma_2,\sigma_1}\right] . \tag{4b}$$

Hence

$$\langle e^{igP_M A_S}\rangle^W_{\binom{\sigma_2}{\mu_2},\binom{\sigma_1}{\mu_1}} \approx e^{ig\langle P_M\rangle^W_{\mu_2,\mu_1}\langle A_S\rangle^W_{\sigma_2,\sigma_1}} . \tag{5}$$

Note that the argument of the exponential on the RHS of the last equation has both a real and an imaginary part. Therefore,

$$\left|\langle e^{igP_M A_S}\rangle^W_{\binom{\sigma_2}{\mu_2},\binom{\sigma_1}{\mu_1}}\right|^2 \approx e^{-2g\mathrm{Im}(\langle P_M\rangle^W_{\mu_2,\mu_1}\langle A_S\rangle^W_{\sigma_2,\sigma_1})} \tag{6a}$$

$$\approx 1 - 2g\mathrm{Im}(\langle P_M\rangle^W_{\mu_2,\mu_1}\langle A_S\rangle^W_{\sigma_2,\sigma_1}) . \tag{6b}$$

We will now let the index μ_2 assume two values $0, 1$. All other indices $(\mu_1, \sigma_1, \sigma_2)$ will be held fixed. For $\mu_2 = 0, 1$, define

$$b_{\mu_2} = 1 - \left|\langle e^{igP_M A_S}\rangle^W_{\binom{\sigma_2}{\mu_2},\binom{\sigma_1}{\mu_1}}\right|^2 , \tag{7}$$

$$z = x + iy = \langle A_S\rangle^W_{\sigma_2,\sigma_1} \tag{8}$$

where x and y are real, and

$$a_{\mu_2} = \langle P_M\rangle^W_{\mu_2,\mu_1} . \tag{9}$$

Then

$$x[2g\mathrm{Im}(a_{\mu_2})] + y[2g\mathrm{Im}(a_{\mu_2} i)] = b_{\mu_2} \tag{10}$$

for $\mu_2 = 0, 1$. We thus have 2 equations and 2 unknowns (x and y). This allows us to solve for x and y, which are the real and imaginary parts of $\langle A_S\rangle^W_{\sigma_2,\sigma_1}$.

Our derivation is only true to first order in g. Thus, it breaks down unless g is smaller than a certain threshold. g must be greater than zero too. This is why. In all our equations from the very beginning, g always appears multiplied times $\langle A_S\rangle^W_{\sigma_2,\sigma_1}$. Hence, when $g = 0$, all mention of $\langle A_S\rangle^W_{\sigma_2,\sigma_1}$ goes away from our equations so we can't use them to solve for $\langle A_S\rangle^W_{\sigma_2,\sigma_1}$. In conclusion, this formalism is only valid for an intermediate range of g values which may be the empty set.