Brief Introduction to Trotter Rescaling, Lie and Suzuki Approximants(by rrtucci)

Suppose $A, B \in \mathbb{C}^{n\times n}$ and $t \in \mathbb{R}$. Define

$$L_1(t) = e^{tA}e^{tB} \; . \tag{1}$$

In this document, we will refer to $L_1(t)$ as the **Lie first-order approximant** of $e^{t(A+B)}$. We call it a first-order approximant because, for small t, according to the Baker-Campbell-Hausdorff expansion[1, 2],

$$L_1(t) = e^{t(A+B)+\frac{t^2}{2}[A,B]+\mathcal{O}(t^3)} = e^{t(A+B)} + \mathcal{O}(t^2) \; . \tag{2}$$

But what if t isn't small? Even when t is not small, one can still use the Lie approximant to approximate $e^{t(A+B)}$. Indeed, if N is a very large integer, then

$$\begin{aligned}
L_1^N(\frac{t}{N}) &= \left(e^{\frac{t}{N}A}e^{\frac{t}{N}B}\right)^N & \text{(3a)}\\
&= \left(e^{\frac{t}{N}(A+B)+\frac{t^2}{2N^2}[A,B]+\mathcal{O}(\frac{t^3}{N^3})}\right)^N & \text{(3b)}\\
&= e^{t(A+B)+\frac{t^2}{2N}[A,B]+\mathcal{O}(\frac{t^3}{N^2})} & \text{(3c)}\\
&= e^{t(A+B)} + \mathcal{O}(\frac{t^2}{N}) \; . & \text{(3d)}
\end{aligned}$$

Henceforth, will refer to this nice trick as a **Trotter rescaling** of an approximant (in this case, the Lie approximant). See Fig.1.

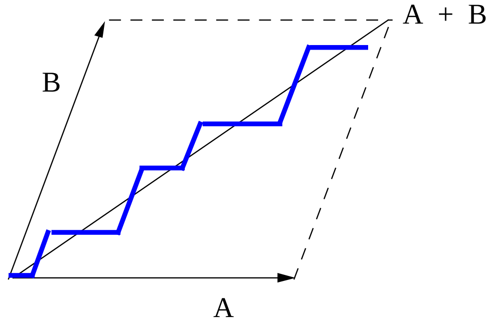

Figure 1: Lie algebra "Physicist's picture" of Trotter rescaling. The system moves from 0 to $A + B$, by moving in small increments in the A and B directions.

Next define

$$S_2(t) = e^{t\frac{A}{2}}e^{tB}e^{t\frac{A}{2}} \; . \tag{4}$$

We will refer to $S_2(t)$ as the **Suzuki second-order approximant** . One can show[1] that for small t:

$$S_2(t) = e^{t(A+B)+\frac{t^3}{6}[\frac{A}{2}+B,[B,A]]+\mathcal{O}(t^5)} = e^{t(A+B)} + \mathcal{O}(t^3) \ . \tag{5}$$

Suzuki[3] also defined higher order approximants based on $S_2(t)$. For $k = 1, 2, 3, \ldots$, define the **Suzuki** $(2k + 2)$**th-order approximant** $S_{2k+2}(t)$ by

$$S_{2k+2}(t) = S_{2k}^2(a_{2k}t)S_{2k}((1 - 4a_{2k})t))S_{2k}^2(a_{2k}t) \ , \tag{6}$$

for some $a_{2k} \in \mathbb{R}$. It is possible to show that for $k = 1, 2, 3, \ldots$ and small t:

$$S_{2k+2}(t) = e^{t(A+B)} + \mathcal{O}(t^{2k+3}) \ \ , \ \text{if} \ \ a_{2k} = \frac{1}{4 - 4^{\frac{1}{2k+1}}} \ . \tag{7}$$

$(a_{2k})_{k=1,2,3,\ldots}$ is a monotone decreasing sequence with $a_2 = 0.4145...$ and $\lim_{k\to\infty} a_{2k} = 1/3$.

As with the Lie approximant, is possible to do a Trotter rescaling of the Suzuki approximants. One finds that for $k = 1, 2, 3 \ldots$, large N and fixed t:

$$S_{2k}^N(\frac{t}{N}) = e^{t(A+B)} + \mathcal{O}(\frac{t^{2k+1}}{N^{2k}}) \ . \tag{8}$$

References

[1] C. Zachos, “Crib Notes on Campbell-Baker-Hausdorff expansions”, http://www.hep.anl.gov/czachos/CBH.pdf

[2] M. Reinsch, “A Simple Expression for the Terms in the Baker-Campbell-Hausdorff Series”, math-ph/9906007

[3] N.Hatano, M.Suzuki, “Finding Exponential Product Formulas of Higher Orders”, arXiv:math-ph/0506007