Honey, I Shrunk the Theory

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1 Introduction

In this blog post, I will give a very brief introduction to Renormalization Group (RG) theory.

This blog is about quantum computing and more generally about quantum information science (QIS). So why should a person working in QIS be interested in RG theory? One reason is that RG theory describes how correlation functions scale, and correlation functions are crucial in: (1)the study of quantum entanglement (2)both classical and quantum Shannon information theory.

Physicists like to study how a theory transforms under a family of operations. Such families of operations usually constitute a mathematical group. The operations might be discrete, as with PTC (P=parity, T=time reversal, C=charge conjugation) or continuous (continuous transformations are a type of generalized rotation). In the case of the renormalization group, physicists consider how a theory transforms under an operation that "scales" the unit of length.

It's useful (at least to me) to think of such scaling as a type of lossy data-compression or smoothing. Accordingly, RG theory can be viewed as a meta-theory that describes how theories change under lossy data-compression. Hence, a more precise but less catchy title for this blog post would have been "Honey, I applied lossy data-compression to the theory (and the kids)."

2 Books and other references on RG

Achtung! There is a huge amount of literature on RG theory, and I'm only familiar with a tiny amount of it, so don't trust too much my advice in this section.

My favorite beginner's book on RG theory is Ref.[1] by McComb. I like this book a lot, although I think many aspects of it could be improved (including fixing all the typos). This book assumes very little prior knowledge, and is very intuitive.

Another book that I consult occasionally is Ref.[2] by Goldenfeld. It's a little more advanced than the book by McComb and covers some topics not covered by McComb.

Last but not least, I recommend Ref.[3] by Srednicki, an excellent, very comprehensive book about quantum field theory from a high energy physics perspective. This book discusses many topics besides the RG, but it includes a nice explanation of Feynman diagrams, renormalization and RG, all from a high energy physics point of view.

Taste in physics books is very subjective so I advise you to inspect the books I've recommend and many others besides. Then form your own opinions.

Besides books, the reference section at the end of this article also lists a few relevant Wikipedia articles that I found interesting.

3 RG theory- a big tent

RG theory is truly a big tent. It has been applied in a wide variety of scientific areas. Here is a partial list of some of those areas. Doubtlessly, I've failed to list some areas, and new areas will arise in the future.

- classical Shannon information theory
- quantum information science, quantum Shannon information theory, study of quantum entanglement. Density Matrix Renormalization Group (DMRG).
- fluid mechanics (turbulence)
- chaos and fractal theory
- differential equations
- statistical mechanics (phase transitions)
- probability and statistics
- quantum field theory, both relativistic (used in high energy physics) and non-relativistic (used in condensed matter physics)
- (as IBM's Watson would say, ???) computational complexity and algorithms theory, how the number of steps in an algorithm depends on the number of input bits n.

4 What is renormalization?

The subject of renormalization methods includes the following topics:

- regularization: defining a divergent integral as the limit of a finite integral. The parameter than one takes the limit of is called the regulator of the integral.
- renormalization: a process of possibly adding a finite number of new interactions to a field theory and dividing certain parameters of the field theory by normalization constants so as to make the field theory finite. These normalization constants are parameterized by a regulator and diverge as that regulator tends to a certain value.
- RG theory: a meta-theory about how a field theory transforms under scale transformations.
- perturbation expansion of a field theory

This article will concentrate on the topic of RG theory, but all these topics are closely related.

Let "**dofs**" stand for degrees of freedom.

Note that the word "scaling" is used in a very special way in RG theory. *Scaling in RG theory = lossy data-compression that reduces the number of dofs of a field theory.* Some synonyms for "scaling" in RG theory: data-compressing, screening, dressing a bare coupling constant, pruning, decimating, averaging, coarse graining, smoothing, forgetting initial conditions, integrating out ultraviolet dofs, zooming out, shrinking. Some antonyms: zooming in, magnifying, expanding, dilating.

Lossy data-compression is associated with the phenomenon of screening, whereby a charged particle polarizes its surrounding medium so that the charge perceived by an observer (called the effective charge) decreases as the observer moves away from the particle.

5 A whiff of thermodynamics

RG theory is intimately linked to thermodynamics, and thermodynamics is a world in itself. Here is but a whiff of thermodynamics. The **partition function** Z is defined by

$$Z = \mathrm{tr}(e^{-\beta H}) = \sum_j e^{-\beta E_j} \ , \tag{1}$$

where $H > 0$ is the Hamiltonian matrix of the system, and $\{E_j\}_j$ are the eigenvalues of H. The **free energy** F is defined in terms of Z by

$$-\beta F = \ln Z \ , \tag{2}$$

where $\beta = \frac{1}{k_B T}$, k_B is Boltzmann's constant, and T is the temperature of the system. All the thermodynamic observables can be expressed in terms of F and it's derivatives with respect to β. McComb calls Eq.(2) the "bridge equation", because it bridges the microscopic physics with the macroscopic one. Nice name! One could write volumes about F alone. However, since this is just a brief article, let's just point out the interesting fact that at low temperatures (i.e., high β), $F \approx E_1$, where E_1 is the lowest (i.e., ground state) energy of the system.

6 An essence of field theory

In RG theory, one considers a "field" (for instance, a scalar field $\phi(x) \in \mathbb{R}$ assigned to each point $x \in \mathbb{R}^d$, or a spin field $S_i \in \{-1, 1\}$ assigned to each point i of a discrete lattice embedded in $\mathbb{R}^d$).

Let d be the number of dimensions of the space that labels the field being considered. For instance, $d = 3$ when the field being considered varies over 3 spatial directions but is stationary (i.e., does not depend on time). In relativistic quantum field theory, one is interested in $d = 4$ because a different field value is attached to each "event" (i.e. space-time point).

As is customary in the RG theory literature, we will use ϵ to denote

$$\epsilon = 4 - d \; . \tag{3}$$

(Mnemonic: note that the above equation doesn't change if you interchange ϵ and d. The equation $\epsilon = d - 4$ (not used here) is not invariant under the same interchange.)

On a cubic lattice, the separation in the x (or y or z) direction between two neighboring lattice points is called the **lattice constant** and will be denoted by a.

Field theories are described by a Hamiltonian. In RG theory, one often considers the Ising Hamiltonian and variants thereof. The Ising Hamiltonian in d dimensions is defined by

$$H = -J \sum_{(i,j) \in \, nn} S_i S_j - B \sum_{i \in \, lp} S_i \; , \tag{4}$$

where

$$\begin{aligned} lp &= \text{ a set of lattice points in a hyper-cubic lattice embedded in } \mathbb{R}^d, \\ nn &= \{(i,j) \in lp : i \text{ and } j \text{ are nearest neighbors}\}, \end{aligned} \tag{5}$$

$S_i \in \{-1, 1\}$ for all $i \in lp$. S_i represents the spin at lattice point i, J is the spin-spin coupling, and B is an external magnetic field interacting with the spins. One often defines generalized coupling constants K_j by $K_1 = \beta J$, $K_2 = \beta B$.

Continuous instead of discrete field theories are also frequently considered in RG theory. For example, the Hamiltonian for the ϕ^4 theory with mass m and coupling constant λ is given by[1]

$$H = \int d^d x \left\{ \frac{1}{2} \sum_{j=1}^{d} (\partial_j \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right\} . \tag{6}$$

One can define generalized coupling constants here too: let $K_1 = m^2$ and $K_2 = \lambda$.

A beautiful aspect of RG theory is that it applies to any field theory, be it continuous or discrete, relativistic (as considered in high energy physics) or non-relativistic (as considered in condensed matter physics).

7 Naive versus fractal "scaling" dimensions

Let the real number $b \geq 1$ denote the **spatial compression (i.e., spatial scaling) factor**. Upon compression, the distance δx between any two spatial positions shrinks to $\delta x' = \frac{\delta x}{b}$.

In high energy physics, it is customary to use an **energy scaling factor** $b_e = 1/b$. It is useful to be fluent in both b and b_e languages. Since $1 \leq b < \infty$, it follows that $0 < b_e \leq 1$. As b goes from 1 to ∞, b_e goes from 1 to 0. b and b_e are both 1 at the same time. They both start at 1, but one goes up and the other down. The point $b = b_e = 1$ corresponds to the microscopic, ultraviolet theory, whereas the point $b = \infty$, $b_e = 0$ corresponds to the macroscopic, infrared theory. Note also that $\ln b_e = -\ln b$ so that as $\ln b$ goes from 0 to ∞, $\ln b_e$ goes from 0 to $-\infty$.

If one uses Planck units in which $\hbar = c = 1$, then time and length have dimensions of L, whereas energy, mass, and momentum have dimensions of $1/L$. For a length x and an energy E, one has

$$x' = \frac{x}{b} = x b_e , \tag{7}$$

$$E' = bE = \frac{E}{b_e} . \tag{8}$$

A physical quantity q can be assigned both a **naive dimension** D_q given by dimensional analysis, and an **anomalous or fractal dimension** y_q. This is expressed with the following notation:

$$[q] = L^{D_q} , \tag{9}$$

and[2]

[1] In equation Eq.(6), ϕ, m, λ represent the bare quantities, often denoted by ϕ_0, m_0, λ_0.

[2] Our definition Eq.(10) of fractal dimension y_q is the standard definition of fractal dimension, namely log of number of self similar pieces divided by log of magnification. See Ref.[5]

$$q' = b^{y_q} q \ . \tag{10}$$

For example, given 3 quantities q_j for $j = 1, 2, 3$, then

$$\left[\frac{q_1}{q_2 q_3}\right] = L^{D_1 - D_2 - D_3} \ , \tag{11}$$

and

$$\frac{q'_1}{q'_2 q'_3} = b^{y_1 - y_2 - y_3} \frac{q_1}{q_2 q_3} \ . \tag{12}$$

Note that a length x has $D_x = 1$ and $y_x = -1$. In some cases, it makes sense to set $y_q = -D_q$ for all quantities q. However, in other cases, one cannot set $D_q = -y_q$ for all quantities q. The cases where one can (ditto, cannot) set $D_q = -y_q$ for all q are associated with lossless (ditto, lossy) data-compression. We try to explain why this is so in Section 9.

8 Real and momentum space RG

Let $\phi_x \in \mathbb{R}$ be the value of a field at position $x \in \mathbb{R}^d$. For each x, ϕ_x is a dof of the theory. One can also consider a conjugate dof ϕ_p which is the Fourier transform of ϕ_x. Only ϕ_p for momenta p in the ball $\{p : |p| < \Lambda\}$ are allowed. Λ, the maximum momentum magnitude allowed, is called the **cutoff momentum**.

In **real space RG** applied to a cubic lattice with lattice constant a, we map a d-dimensional block with sides of length ba into a block with sides of length a. In so doing, we are data-compressing the information assigned to $(ba)^d$ lattice points into the information assigned to a single lattice point. (The information assigned to a lattice point is the value of ϕ_x or other field attached to that lattice point.)

In **momentum space RG**, we data-compress by mapping the information in $\{\phi_p : |p| < \Lambda\}$ into the information in $\{\phi_p : |p| < \Lambda/b\}$. This process is often described by saying that we are integrating the ultraviolet dofs of the theory.

So what do real and momentum space RG have in common? In both real and momentum space RG, we reduce the number of dofs of the theory being considered from N_{dofs} to N'_{dofs}, where

$$b = \frac{N_{dofs}}{N'_{dofs}} \ . \tag{13}$$

9 Correlations rule the world

As before, let $\phi_x \in \mathbb{R}$ be the value of a field at position $x \in \mathbb{R}^d$. ϕ_x is taken to be a random variable. How the average $\langle \phi_{x_1} \phi_{x_2} \rangle$ decays with distance $r = |x_1 - x_2|$ is characterized by a decay length ξ called the **correlation length** of ϕ_x.

Lossless (ditto, lossy) data-compression of the theory is associated with zero (ditto, non-zero) ξ. The intuition for this fact is that a field with $\xi = 0$ is composed of probabilistically independent dofs ϕ_x, and therefore pruning those dofs causes no loss of information.

As mentioned before, in the case of lossless (ditto, lossy) data-compression, one can (ditto, cannot) set the naive and fractal dimensions equal (up to a sign) for every physical quantity (i.e., $y_q = -D_q$ for all q).

One can prove that there is a number d_{uc} called the **upper critical dimension**, such that $\xi = 0$ if $d \geq d_{uc}$. For instance, for a ϕ^4 theory, $d_{uc} = 4$. For $d \geq d_{uc}$, one can "neglect the thermal fluctuations", set $\xi = 0$, assume lossless data-compression, and set $y_q = -D_q$ for all q. The Landau and mean field models describe this lossless regime.

10 Renormalization (semi-)group

Let $b, b_1, b_2, b_3 \geq 1$. If R_b denotes data-compression with compression factor b of a theory, then clearly

$$R_{b_1 b_2} = R_{b_1} R_{b_2} \ . \tag{14}$$

Note that $R_1 R_b = R_b$ (identity property), and $R_{b_1}(R_{b_2} R_{b_3}) = (R_{b_1} R_{b_2}) R_{b_3}$ (associative property). However, one cannot set $R_{1/b} R_b = R_1$ (inverse property) because $1/b$ is not greater than 1. Physically, the reason RG operations don't have inverses is that they represent lossy data-compression, and therefore are irreversible. So the set of transformation $\{R_b : b \geq 1\}$ is not a true mathematical group. In truth, it only satisfies the properties of a weaker algebraic structure called a semi-group.[3]. Physicists, however, call it a group anyway.

11 RG streamlines

A theory is described by a Hamiltonian H. Each interaction term in the Hamiltonian is multiplied by a different **generalized c.c. (coupling constant)** K_j which measures the strength of that interaction term. Let $\vec{K} = (K_1, K_2, \ldots K_{N_{c.c.}})$. Applying R_b once gives

$$H' = R_b H, \ \ \vec{K}' = R_b \vec{K} \ . \tag{15}$$

Applying R_b multiple times gives

$$H^{(n+1)} = R_b H^{(n)}, \ \ \vec{K}^{(n+1)} = R_b \vec{K}^{(n)} \ , \tag{16}$$

[3]However, if the theory is scale invariant as happens when it sits at a critical point, then one can consider all $b > 0$, not just $b \geq 1$, and the set $\{R_b : b > 0\}$ is a true group.

$$H^{(n)} = R_b^n H^{(0)}, \;\; \vec{K}^{(n)} = R_b^n \vec{K}^{(0)} \;, \tag{17}$$

for $n = 0, 1, 2, \ldots$. The sequence $\vec{K}^{(0)}, \vec{K}^{(1)}, \vec{K}^{(2)}, \ldots$ describes a one-dimensional curve (a streamline, a flow) in c.c. space. A different curve is traced for each initial condition $\vec{K}^{(0)}$. The streamlines flow from $b = 1$ to $b = \infty$ (or, in b_e language, from $b_e = 1$ to $b_e = 0$).

12 Fixed points of the trivial and critical kind

If, as n tends to infinity,

$$\vec{K}^{(n)} \to \vec{K}^* \;, \tag{18}$$

we say $\vec{K}^*$ is a **fixed point in the c.c.'s**. Other functions of $\vec{K}$ like the Hamiltonian $H(\vec{K})$ and the correlation length $\xi(\vec{K})$ also tend to a fixed point

$$H(\vec{K}^*) = H^*, \;\; \xi(\vec{K}^*) = \xi^* \;. \tag{19}$$

Note that the correlation length ξ transforms as

$$\xi' = \frac{\xi}{b} \;, \;\; \xi^{(n)} = \frac{\xi^{(0)}}{b^n} \;. \tag{20}$$

Therefore, for large n,

$$\xi^* = \frac{\xi^*}{b^n} \;. \tag{21}$$

Eq.(21) has two possible solutions. Either $\xi^* = 0$, in which case we say the fixed point $\vec{K}^*$ is trivial, or $\xi^* = \infty$, in which case we say the fixed point is critical.

Trivial fixed points are not very interesting. Since they have $\xi^* = 0$, they can be understood in terms of lossless data-compression and naive dimensions for the c.c.'s.

Critical points are much more interesting. Since they have $\xi^* = \infty$, they can be understood in terms of lossy data-compression and fractal dimensions for the c.c.'s.

Two examples of critical points that occur in nature are the Curie temperature of an Ising system (the temperature at which the magnetization of the system goes from zero to growing, spontaneously, at zero externally applied magnetic field). Another showcase critical point is the critical point of water (a unique point in the PVT phase diagram of water, a point at which gaseous and liquid water coexist). The temperature at which a critical point occurs is called its **critical temperature** and is denoted by T_c. Critical points are also sometimes called second order phase transitions.

13 Critical exponents and universality classes

How various thermodynamic quantities behave near and precisely at a critical point is described by a slew of **critical exponents**. It is common to define 6 critical exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$ (all defined so as to be non-negative) defined on page 150 of McComb. One can use very general thermodynamic arguments to prove that these 6 exponents must satisfy 4 simple relationships between them. Thus, one need only calculate 2 of 6. The 2 that are usually calculated are ν and η.

ν and η can both be inferred from the behavior of Greens's functions. Let ϕ_x be the value of a field at point x. Let $\langle Q \rangle$ denote the average of a quantity Q. We will use the following shorthand notations

$$r = |x_1 - x_2|, \;\; \Delta\phi_x = \phi_x - \langle \phi_x \rangle, \;\; \theta_c = \frac{\Delta T}{T_c} = \frac{T - T_c}{T_c} \; . \tag{22}$$

We define the **Greens's function** $G(r)$ by

$$G(r) = \langle \phi_{x_1} \phi_{x_2} \rangle \; , \tag{23}$$

and the **connected Greens's function** $G_c(r)$ by

$$G_c(r) = \langle \Delta\phi_{x_1} \;\; \Delta\phi_{x_2} \rangle = \langle \phi_{x_1} \phi_{x_2} \rangle - \langle \phi_{x_1} \rangle \langle \phi_{x_2} \rangle = \langle \phi_{x_1}, \phi_{x_2} \rangle \; . \tag{24}$$

The behavior of $G_c(r)$ near but not precisely at a critical point is of an exponential form, with decay constant give by the correlation length ξ. The dependance of ξ on temperature determines the critical exponent ν:

$$G_c(r) \sim \exp(-r/\xi), \;\; \xi = |\theta_c|^{-\nu} \; . \tag{25}$$

(Note that for $\nu > 0$, $\xi \to \infty$ as $T \to T_c$.) The behavior of $G_c(r)$ precisely at a critical point is an inverse power law, and the power of the law determines the critical exponent η:

$$G_c(r) \sim 1/r^{pow}, \;\; pow = d - 2 + \eta \; . \tag{26}$$

(Let $d = 4$. If $\eta = 0$, this gives an inverse square law. If $\eta > 0$, $G_c(r)$ decays faster than when $\eta = 0$, as one expects if there is screening.)

For all field theories, all critical points with the same critical exponents are said to be in the same **universality class**. For example, the ϕ^4 theory has a single trivial fixed point for $d \geq 4$, but for $d \leq 4$, it has a single critical fixed point, called the Wilson-Fisher point, which has the same critical exponents as the critical point of the Ising model in d dimensions.

Universality classes are few and far between because theories are scale invariant at a critical point, and scale invariant theories are few and far between.

14 Relevant, marginal and irrelevant operators

It is interesting to analyze the behavior of a field theory in the vicinity of a critical point. To do this, it is convenient to replace the generalized c.c.'s $\vec{K} = (K_1, K_2, \ldots, K_{N_{c.c.}})$ by **natural c.c.'s** $\vec{g} = (g_1, g_2, \ldots, g_{N_{c.c.}})$. The natural c.c.'s are natural coordinates for describing the particular critical point being considered. They have the critical point as their origin (i.e., $\vec{g^*} = 0$).

The set of all points that flow into the critical point (i.e., the "basin of attraction" of the critical point) is called the **critical surface**. As a consequence of Eq.(20), all points in the critical surface have infinite correlation length ξ, just like the critical point itself (in a manner of speaking, they get infected by the critical point).

Note that

$$g'_j = b^{y_j} g_j, \quad g_j^{(n+1)} = b^{ny_j} g_j^{(0)} \tag{27}$$

for all j. Since $b \geq 1$, there are three possible scenarios for each g_j:

- $y_j > 1$: In this case $g_j^{(n)} \to \infty$ as $n \to \infty$. We say g_j is a relevant c.c..
- $y_j < 1$: In this case $g_j^{(n)} \to 0$ as $n \to \infty$. We say g_j is an irrelevant c.c..
- $y_j = 1$: In this case $g_j^{(n)}$ tends to a constant as $n \to \infty$. We say g_j is a marginal c.c..

If an interaction term in the Hamiltonian is multiplied by a relevant c.c., we call that interaction term a relevant operator, and likewise for irrelevant and marginal.

Fig.1 shows the typical behavior of the RG streamlines in the vicinity of a critical point, assuming $N_{c.c.} = 2$. In Fig.1, point C is the critical point. The K_1, K_2 axes are for the generalized c.c.'s and the g_1, g_2 axes are for the natural c.c.'s. The arrows on the streamlines point in the direction of increasing b (i.e., decreasing b_e). In the case of Fig.1, the critical surface is a one dimensional curve, the one that contains points S, C and S'. No streamlines ever cross the critical surface. They stay for all b either above or below the critical surface.

If the initial conditions are such that the theory starts somewhere on the critical surface, then for b large enough, the theory ends at point C where $g_1 = g_2 = 0$. On the other hand, if the theory starts at a point such as A which is close to the critical surface but not on it, then the irrelevant c.c. g_1 (ditto, relevant c.c. g_2) shrinks (ditto, grows) monotonically as b grows. By the time the theory reaches, say, point B, g_1 is negligible, and g_2 is starting to become uncomfortably large for expanding in powers of it. A good place to do perturbation theory is in the vicinity of point P, where the irrelevant c.c. g_1 is negligible, and the relevant c.c g_2 is much smaller than 1, but still non-zero.

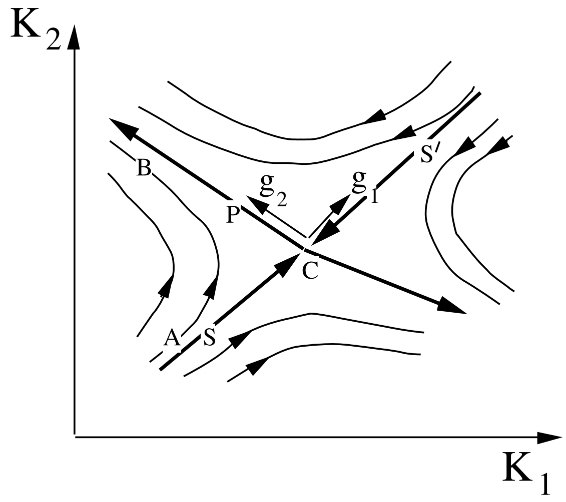

Figure 1: RG streamlines in the vicinity of a critical point for a theory with 2 coupling constants.

15 Self-similar coupling constants and beta functions

Consider a critical point with natural c.c.'s $\vec{g} = (g_1, g_2, \ldots, g_{N_{c.c.}})$. Define the fractal dimension y_j of the c.c. g_j by

$$\frac{d \ln g_j}{d \ln b} = y_j \ . \tag{28}$$

In high energy physics, it is common to use, instead of y_j, a so called **beta function** β_j defined by

$$\frac{d g_j}{d \ln b_e} = \beta_j \ . \tag{29}$$

Comparing of Eqs.(28) and (29) and using $\ln b = -\ln b_e$, one finds

$$\beta_j = -g_j \, y_j \ . \tag{30}$$

Precisely at the critical point (i.e., at $\vec{g} = 0$), one must have $\beta_j = 0$. Amazingly, in the vicinity of the critical point, the beta function β_j must be of the form[4]

[4]When $\epsilon = 0$, the expansion of the beta function in powers of g_j usually starts at order(g_j^2). This is why. The value of a_{j1} (i.e., the order(g_j^1) coefficient in the expansion of the beta function in powers of g_j) must equal the naive dimension of g_j. As explained in Section 16, one usually introduces a renormalization-point mass μ to make $[g_j] = L^0$. If this is done, a_{j1} becomes proportional to ϵ, and vanishes when $\epsilon = 0$.

$$\beta_j = \beta_j(g_j, \epsilon) = a_{j1}(\epsilon) g_j + a_{j2}(\epsilon) g_j^2 + a_{j3}(\epsilon) g_j^3 + \cdots \ , \tag{31}$$

where the coefficients a_{jk} are functions only of ϵ, not of b_e. Furthermore, one must have

$$\lim_{\epsilon \to 0} \beta_j = \text{ finite number } . \tag{32}$$

Eqs.(29) and (31) yield an easy to solve, separable differential equation. Hence, given g_j and $\ln(b'_e/b_e)$, one can find the value of g'_j. One describes this felicitous property by saying that the c.c. g_j is self-similar or by saying the humorous statement that g_j is a "running c.c.".

More generally, if the theory has a set of generalized c.c.'s $\vec{K} = (K_1, K_2, \ldots, K_{N_{c.c.}})$ that are not associated with a particular critical point, one can still define beta functions $\beta_j(b_e)$ by

$$\frac{dK_j}{d \ln b_e} = \beta_j(b_e) \ . \tag{33}$$

Then a fixed point in the c.c.'s is defined as any point $\vec{K}^*$ at which $\beta_j = 0$ for all j. If $\vec{K}^*$ occurs when $b_e = 0$ (ditto, $b_e = 1$), we call it an infrared (ditto, ultraviolet) fixed point.[5] For instance, QED (Quantum Electrodynamics) has $\beta > 0$ which gives it an infrared fixed point, whereas QCD (Quantum Chromodynamics) has $\beta < 0$ which gives it an ultraviolet fixed point.

16 The regulator and the fiducial mass scale

A continuous quantum field theory has infinite terms in its perturbation expansion. That's not surprising because such a theory describes infinitely many harmonic oscillators, and each of those harmonic oscillators has a finite zero-point energy of $\hbar\omega/2$. One regulates the infinities of a quantum field theory by introducing a **regulator** parameter with a "catastrophic" value. The infinities go away if the regulator is slightly off from its catastrophic value, and they come back as the regulator tends to its catastrophic value. All observable quantities must be independent of the regulator. The two most popular regulators in the high energy physics literature are

1. a momentum cutoff Λ. This is called Pauli-Villars regularization.

2. $\epsilon = 4 - d$. This is called dimensional regularization.

[5]Some might be puzzled by the fact that we speak of b_e as if it were an energy, although, in reality, it is a *ratio* of energies. If the theory has a lattice constant a, define $x_{min} = a$ and $E_{max} = 1/a$. Recall that $b_e = E/E'$. If we keep E' fixed at $E' = E_{max}$, then b_e and E are proportional to each other.

In case (1), the space-time dimension d is kept fixed at an integer value, whereas in (2), d varies over a continuum of real numbers.

In case (2), in order to make the c.c.'s dimensionless, one needs to introduce a **fiducial mass scale**.[6] This fiducial mass scale, called the **renormalization-point**, is usually denoted by μ.[7] In both cases (1) and (2), we need to introduce a fiducial mass scale, Λ in case (1) and μ in case (2). Roughly speaking, the fiducial mass scale parameterizes how we subtract the infinite part of the answer. In case (1), Λ acts both as the regulator parameter and the fiducial mass scale. In case (2), these two roles are assigned to separate parameters, ϵ becomes the regulator and μ the fiducial mass scale.

High energy physicists usually do not use a scaling factor like b_e or b. Instead, they use $\ln \mu$ (or $\ln \Lambda$) en lieu of $\ln b_e$ in Eq.(33). That's no problem. Just replace μ by $b_e\mu$ (or Λ by $b_e\Lambda$). If you keep μ (or Λ) fixed and allow b_e to vary, then $d\ln(b_e\mu)$ (or $d\ln(b_e\Lambda)$) can be replaced by $d\ln b_e$.

17 RG theory has its pi-groups too! Callan-Symanzik Type Equations

In doing dimensional analysis (Ref.[6]) for a given problem, one usually begins by finding the pi-groups relevant to the problem. A pi-group (so called because the symbols $\pi_1, \pi_2, \ldots$ are used to represent them) is a product of quantities that is dimensionless:

$$[\text{dim-analysis pi-group}] = L^0 \ . \tag{34}$$

Analogously, in RG theory, one can often identify certain compound quantities which must be scale invariant (i.e., independent of b_e). I like to call such quantities RG pi-groups.

$$\frac{d}{d\ln b_e}\{\text{RG pi-group}\} = 0 \ . \tag{35}$$

The Callan-Symanzik equation used in high energy physics is just a special case of Eq.(35).

[6] That's my own name for it, invented after reading a credit card contract.

[7] For those who have studied quantum field theory previously, let me remind you of how the renormalization-point μ arises. The c.c. g is replaced by $\mu^p g$ where the power p is chosen so as to make g dimensionless. $[(d^dx)(\partial\phi)^2] = L^0$ implies $[\phi] = L^{1-\frac{d}{2}}$. Then $[(d^dx)\mu^p(g\phi^n)] = L^0$ with $[g] = L^0$ and $[\mu] = L^{-1}$ implies $d - p + n(1-\frac{d}{2}) = 0$. Hence, $p = d(1-\frac{n}{2}) + n$. For example, in the ϕ^3 theory in 6 dimensions considered in Srednicki's book Ref.[3], one gets, using $d = 6-\epsilon$, that $p = -(6-\epsilon)\frac{1}{2} + 3 = \frac{\epsilon}{2}$.

18 Forgetting initial conditions. Are we cheating with infinities? Where did the infinities go?

Sometimes it is useful to have infinities in the initial conditions of a problem. For example, it's useful to use a Dirac delta function as initial conditions for a diffusion equation (or for the differential equation obeyed by a Greens's function). The infinities in the initial conditions are " forgotten" at later times, as the solution spreads.

A similar situation occurs with quantum field theories. One starts with bare c.c.'s like ϕ_0, m_0, λ_0 and these evolve to dressed c.c.'s like ϕ, m, λ. The bare c.c.'s correspond to $\vec{K}^{(0)}$ at ultra-high energy scales $b_e = b = 1$. The dressed c.c.'s correspond to $\vec{K}^{(n)}$ for a large n and not-so-ultra-high energy scales, $b_e \in (0, 1)$ but $b_e << 1$. The bare c.c.'s are infinite, but never mind, they are quickly forgotten. They are forgotten because the c.c.'s are self-similar, so we can find g' given g and $\ln(b'_e/b_e)$. If g is set to its bare value $g = g_0$, then g and therefore g' will be infinite, but we never set $g = g_0$.

Note that by the same token, any theory which has infinite initial conditions which are forgotten, can be regulated with a regulator, and a fiducial mass scale, call it μ. Then μ can be replaced by $b_e\mu$ with $b_e \in (0, 1)$, and RG theory can be used. The upshot is that any theory with infinite initial conditions which are forgotten is amenable to analysis via RG theory.

19 The many faces of a renormalizable theory

When high energy physicists talk about a **renormalizable theory**, their definition of what that means keeps changing. Several popular definitions are

1. A theory that yields finite predictions for the observables when those observables are expressed as perturbation expansions in powers of the c.c.'s.

2. A theory for which only a finite number of c.c.'s are needed in order to achieve finite predictions for the observables.

3. A theory for which the c.c.'s are self-similar.

4. A theory for which the c.c.'s are dimensionless when $\epsilon = 4 - d = 0$. (Otherwise, cross-sections blow up at high energies).

So which one is it? One great insight of RG theory is that all these definitions are roughly equivalent. These properties are all symptoms of a field theory that is in the vicinity of a critical point.

References

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[5] http://en.wikipedia.org/wiki/Fractal_dimension

[6] http://en.wikipedia.org/wiki/Dimensional_analysis

[7] http://en.wikipedia.org/wiki/Landau_pole

[8] http://en.wikipedia.org/wiki/Scale_invariance