# EPR, 2 Particles, Theory

This section deals with the theory of EPR experiments in which 2 spin 1/2 fermions fly apart. We will discuss two variations of this experiment. These variations will be referred to as the Bohm-Bell and the Clauser-Horne experiments.

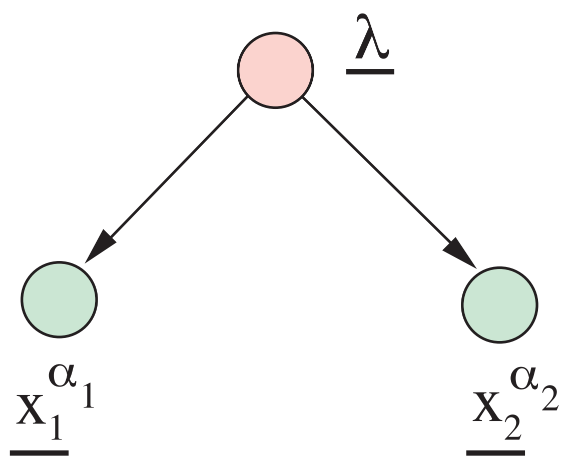


FIG.1

In local realistic theories, an EPR experiment in which 2 spin 1/2 fermions fly apart is described by the classical Bayesian net shown in Fig.1. In this figure, node $\underline{\lambda}$ represents the *hidden variables*. We will call $\Lambda$ the set of states $\lambda$ which node $\underline{\lambda}$ can assume. For $j \in \{1, 2\}$, node $\underline{x_j^{\alpha_j}}$ represents the outcome of a spin measurement performed on particle $j$. $\alpha_j$ represents the measurement axis. Node $\underline{x_j^{\alpha_j}}$ may assume two possible states, + or −, depending on whether the measurement finds the spin to be pointing up or down along the $\alpha_j$ axis. For example, $\underline{x_1^A} = +$ if a measurement of the spin of particle 1 along the $A$ axis yields "up".

It is convenient to define probability functions $P_j^{\alpha_j}(\cdot|\cdot)$, $P_j^{\alpha_j}(\cdot)$, $P_{12}^{\alpha_1\alpha_2}(\cdot|\cdot)$ and $P_{12}^{\alpha_1\alpha_2}(\cdot)$ as follows:

$$P_j^{\alpha_j}(x_j|\lambda) = P(\underline{x_j^{\alpha_j}} = x_j|\underline{\lambda} = \lambda) , \tag{1}$$

$$P_j^{\alpha_j}(x_j) = P(\underline{x_j^{\alpha_j}} = x_j) , \tag{2}$$

$$P_{12}^{\alpha_1\alpha_2}(x_1, x_2|\lambda) = P(\underline{x_1^{\alpha_1}} = x_1, \underline{x_2^{\alpha_2}} = x_2|\underline{\lambda} = \lambda) , \tag{3}$$

$$P_{12}^{\alpha_1\alpha_2}(x_1, x_2) = P(\underline{x_1^{\alpha_1}} = x_1, \underline{x_2^{\alpha_2}} = x_2) \ , \tag{4}$$

where $j \in \{1, 2\}$.

Fig.1 implies the following equation:

$$P_{12}^{\alpha_1\alpha_2}(x_1, x_2) = \sum_{\lambda\in\Lambda} P_1^{\alpha_1}(x_1|\lambda)P_2^{\alpha_2}(x_2|\lambda)P(\lambda) \ . \tag{5}$$

Because they satisfy Eq.(5), the random variables $\underline{x_1^{\alpha_1}}$ and $\underline{x_2^{\alpha_2}}$ are said to be *conditionally independent* (with respect to $\underline{\lambda}$). Note that conditionally independent variables $\underline{x_1^{\alpha_1}}$ and $\underline{x_2^{\alpha_2}}$ become independent (independent in the sense of probability theory) if the value of $\lambda$ is fixed by setting $P(\lambda) = \delta(\lambda, \lambda_0)$. The acts of measuring $\underline{x_1^{\alpha_1}}$ and $\underline{x_2^{\alpha_2}}$ constitute two events. If the separation between these 2 events is spacelike, then local realistic theories require that Eq.(5) be true.

We will assume that the particles are created in a state of zero total spin angular momentum, and that they then fly apart without interacting with anything else. In Quantum Mechanics, this means that the particles are in the antisymmetric, singlet state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+_z -_z\rangle - |-_z +_z\rangle) \ . \tag{6}$$

One can show (see the appendix entitled "Spin 1/2 Particles") that for this state,

$$P_{12}^{\alpha_1\alpha_2}(++) = P_{12}^{\alpha_1\alpha_2}(--) = \frac{1}{2}\sin^2(\frac{\angle(\alpha_1, \alpha_2)}{2}) \ , \tag{7a}$$

$$P_{12}^{\alpha_1\alpha_2}(+-) = P_{12}^{\alpha_1\alpha_2}(-+) = \frac{1}{2}\cos^2(\frac{\angle(\alpha_1, \alpha_2)}{2}) \ , \tag{7b}$$

$$P_1^{\alpha_1}(+) = P_2^{\alpha_2}(+) = \frac{1}{2} \ , \tag{7c}$$

where $\angle(\alpha_1, \alpha_2)$ is the angle between axes $\alpha_1$ and $\alpha_2$.

# Bohm-Bell Experiment

In the Bohm-Bell experiment, the spin of both particles is measured along the same 3 axes. Thus, if we call the directions of these axes $A, B$ and $C$, then $\alpha_1, \alpha_2 \in \{A, B, C\}$. Suppose $x, y$ and $z$ are either $+$ or $-$. The Bell-inequalities for the Bohm-Bell experiment are:

$$P_{12}^{AC}(x, z) \leq P_{12}^{AB}(x, y) + P_{12}^{BC}(y, z) \ , \tag{8}$$

and the 5 other inequalities one gets by permuting the symbols $A, B$ and $C$. Eq.(8) is proven in an appendix at the end of this section. The proof given in the appendix assumes that Local Realism holds.

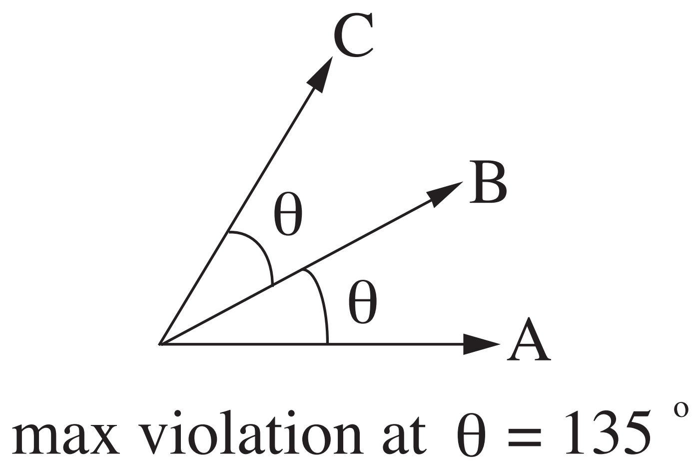


FIG. 2

Next we will combine the local realistic result Eq.(8) with the quantum mechanical results Eqs.(7) and arrive at a contradiction. Assume axes $A, B$ and $C$ are coplanar and $\angle(A, B) = \angle(B, C) = \theta$ (see Fig.2). Also let $x = +, y = -$ and $z = +$ in Eq.(8). Then combining Eq.(8) with Eqs.(7) yields

$$\frac{1}{2}\sin^2(\theta) \leq \frac{1}{2}\cos^2(\frac{\theta}{2}) + \frac{1}{2}\cos^2(\frac{\theta}{2}) \ . \tag{9}$$

This inequality can be simplified to

$$0 \leq 1 + \cos(2\theta) + 2\cos(\theta) \ , \tag{10}$$

which is violated (maximally) when $\theta = \frac{3\pi}{4} = 135^o$.

Thus, Quantum Mechanics tells you that if you measure the spin of particle 1 along the A axis and the spin of 2 along C, where angle(A, C) = 270 degs., and if you do this many times, you will get a probability $P_{12}^{AC}(+,+)$ that is greater than predicted by Local Realism. Somehow the particles know more about each other than one would have expected from Local Realism alone.

## Clauser-Horne Experiment

In the Clauser-Horne experiment, the spin of particle 1 is measured along axes $A$ and $A'$ and that of particle 2 along axes $B$ and $B'$. Thus, $\alpha_1 \in \{A, A'\}$ and $\alpha_2 \in \{B, B'\}$. The Bell inequalities for the Clauser-Horne experiment are:

$$0 \leq 1 + P_{12}^{AB}(++) + P_{12}^{A'B}(++) + P_{12}^{AB'}(++) - P_{12}^{A'B'}(++) - P_1^A(+) - P_2^B(+) \leq 1 \ , \tag{11}$$

and the three other inequalities produced by (1)interchanging $A$ with $A'$, (2)interchanging $B$ with $B'$, (3) interchanging $A$ with $A'$, and $B$ with $B'$. We won't present any proof of Eq.(11) here. It may be proven in various ways. See Refs.[1]-[3] if interested. The proofs given in those references assume that Local Realism holds.

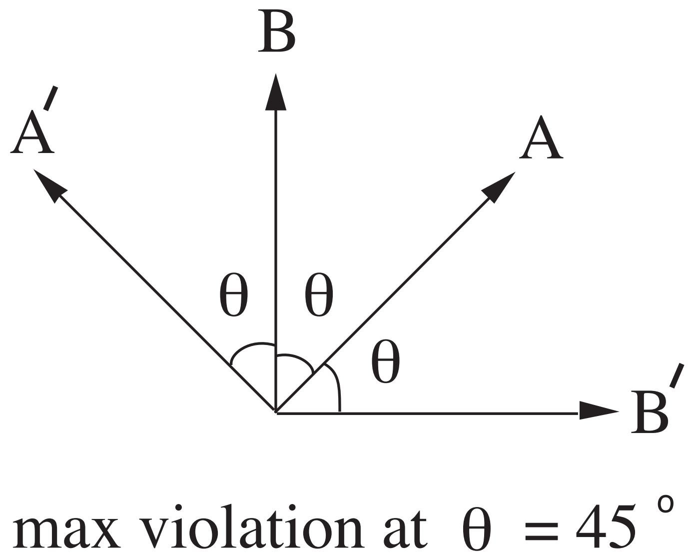


FIG. 3

Next we will combine the local realistic result Eq.(11) with the quantum mechanical results Eqs.(7) to arrive at a contradiction. Assume axes $A, A', B$ and $B'$ are coplanar and $\angle(B', A) = \angle(A, B) = \angle(B, A') = \theta$ (see Fig.3). Then combining Eq.(11) with Eqs.(7) yields

$$0 \leq 1 + \frac{3}{2}\sin^2(\frac{\theta}{2}) - \frac{1}{2}\sin^2(\frac{3\theta}{2}) - \frac{1}{2} - \frac{1}{2} \leq 1 \ . \tag{12}$$

This last equation simplifies to

$$-2 \leq \cos(3\theta) - 3\cos(\theta) \leq 2 \ , \tag{13}$$

which is violated (maximally) when $\theta = \frac{\pi}{4} = 45^o$.

**Appendix: Proof Of Bell Inequalities For Bohm-Bell Experiment**

# References

[1] J.F. Clauser and M.A. Horne, Phys. Rev. D **10**, 526 (1974).

[2] Arthur Fine, Phys. Rev. Let. **48**, 291 (1982).

[3] R.R. Tucci, Int. Jour. of Mod. Phys. B **9**, 819 (1995).

# Appendix: Proof Of Bell Inequalities For Bohm-Bell Experiment

This section will present 2 proofs of the Bell Inequalities for the Bohm-Bell experiment.

For $x \in \{+, -\}$, let $\overline{x} = +$ if $x = -$, and vice versa. Hence, $\overline{x}$ is the opposite of $x$.

**Proof 1:** We begin by noticing that since the initial state must have zero spin angular momentum, one must have

$$P_1^\alpha(x|\lambda) = P_2^\alpha(\overline{x}|\lambda) \ , \tag{1}$$

where $\alpha \in \{A, B, C\}$ and $x \in \{+, -\}$. In other words, if we measure the spin of both particles along the same axis, we expect that the two measurements will always be opposite. This should be true in any theory that conserves angular momentum.

One has

$$\begin{array}{l} P_{12}^{AC}(x, z|\lambda) \\ = P_1^A(x|\lambda)P_2^C(z|\lambda) \\ = P_1^A(x|\lambda)[P_1^B(\overline{y}|\lambda) + P_1^B(y|\lambda)]P_1^C(\overline{z}|\lambda) \\ \leq \\ P_1^A(x|\lambda)P_1^B(\overline{y}|\lambda) + P_1^B(y|\lambda)P_1^C(\overline{z}|\lambda) \\ = P_{12}^{AB}(x, y|\lambda) + P_{12}^{BC}(y, z|\lambda) \end{array} \ , \tag{2}$$

where Eq.(1) has been used repeatedly. Multiplying both sides of inequality Eq.(2) by $P(\lambda)$ and adding over all $\lambda \in \Lambda$ yields

$$P_{12}^{AC}(x, z) \leq P_{12}^{AB}(x, y) + P_{12}^{BC}(y, z) \ . \tag{3}$$

**Proof 2:** This proof is based on the following equation from the section entitled "EPR, 2 Particles, Theory":

$$P_{12}^{\alpha_1\alpha_2}(x_1, x_2) = \sum_{\lambda \in \Lambda} P_1^{\alpha_1}(x_1|\lambda)P_2^{\alpha_2}(x_2|\lambda)P(\lambda) \ . \tag{4}$$

Suppose $\underline{x_j^\alpha}$ is a random variable with values $x_j^\alpha \in \{+, -\}$, where $j \in \{1, 2\}$ and $\alpha \in \{A, B, C\}$. $\underline{x_j^\alpha}$ represents the value obtained by a measurement of particle $j$ along axis $\alpha$. Define $\underline{X}_1, \underline{X}_2$ and $\underline{X}$ by

$$\underline{X}_1 = (\underline{x}_1^A, \underline{x}_1^B, \underline{x}_1^C) \ , \ \ \underline{X}_2 = (\underline{x}_2^A, \underline{x}_2^B, \underline{x}_2^C) \ , \ \ \underline{X} = (\underline{X}_1, \underline{X}_2) \ , \tag{5}$$

and define $\underline{X}_1, \underline{X}_2$ and $\underline{X}$ analogously.

Suppose we replace in Eq.(5) the hidden variables $\underline{\lambda}$ by the special hidden variables $\underline{X}$:

$$P(\underline{\lambda} = \lambda) \to P(\underline{X} = X) \ . \tag{6}$$

According to Quantum Mechanics, the probability distribution $P(X)$ does not exist, because its existence would imply that one can know precisely and simultaneous the values of complementary variables such as $x_1^A$ and $x_1^B$. However, Local Realism, which is what we are assuming in this proof, does not object to the existence of $P(X)$.

Since the variables $X$ arise so naturally in this problem, we will call them the *canonical hidden variables* for this problem. It might seem that we loose generality by considering only canonical hidden variables, but this is not so. When the hidden variables are not the canonical ones, their effect on this particular problem can always be mimicked identically by a suitable probability distribution $P(X)$ of the canonical hidden variables.

Notice that because of conservation of angular momentum, $P(X)$ vanishes unless $X_2 = -X_1$. Therefore, $P(X)$ can be expressed as

$$P(X) = \sigma(X_1)\delta(X_1, -X_2) \ , \tag{7}$$

where $\sigma(\cdot)$ is some probability function of $X_1$.

Combining Eq.(7) and Eq.(5), one gets

$$P_{12}^{AB}(x, y) \leq \sum_{x_1^C \in \{+,-\}} \sigma(x, \overline{y}, x_1^C) = \sigma(x, \overline{y}, \overline{z}) + \sigma(x, \overline{y}, z) \ , \tag{8a}$$

$$P_{12}^{BC}(y, z) \leq \sum_{x_1^A \in \{+,-\}} \sigma(x_1^A, y, \overline{z}) = \sigma(x, y, \overline{z}) + \sigma(\overline{x}, y, \overline{z}) \ , \tag{8b}$$

$$P_{12}^{AC}(x, z) \leq \sum_{x_1^B \in \{+,-\}} \sigma(x, x_1^B, \overline{z}) = \sigma(x, y, \overline{z}) + \sigma(x, \overline{y}, \overline{z}) \ . \tag{8c}$$

The first term on the right side of Eq.(8c) is the first term on the right side of Eq.(8b). The second term on the right side of Eq.(8c) is the first term on the right side of Eq.(8a). Therefore, Eq.(3) above follows.

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