Quantum Fog Library Of Essays

written by: Robert R. Tucci

last changed: Sept. 1, 1998

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Introduction

This document discusses each of the nets contained in the folder entitled "QFogLibOfNets".

We've tried to make this document interesting and useful to all members of a very wide audience, including advanced high school students and persons with graduate degrees in Physics. We've tried to include in it something for everyone. Our discussion of each net is fairly complete, even by the standards of any expert in the field of Quantum Mechanics. In order to make these discussions complete, we found it necessary to include mathematical formulas. If you can't follow the math, or if you don't care about it, just skip it. You most definitely do NOT need to understand it in order to develop an understanding of a net. Do try to understand the general aspects of what is being said about a net. Then go and play with the net. Quantum Fog does the math for you. Hence, even if you don't understand the math too well, you can still get correct, numerically precise answers to your questions.

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Mathematical Notation

- $\mathbf{M^{-1}}$ *Inverse* of matrix M.
- $\mathbf{M^{*}}$ *Complex conjugate* of matrix M.
- $\mathbf{M^{T}}$ *Transpose* of matrix M.
- $\mathbf{M^{\dagger}}$ *Adjoint or Hermitian conjugate* of matrix M. It equals $(M^{*})^{T}$. M is a *unitary matrix* iff $MM^{\dagger} = M^{\dagger}M = 1$. M is a *rotation (also called an orthogonal matrix)* iff M is unitary and its entries are all real.
- $[\mathbf{A}, \mathbf{B}]$ For two square matrices A and B of the same size, $[A, B] = AB - BA$ is called their *commutator*. When $[A, B] = 0$, we say that A and B commute.
- $[\mathbf{A}, \mathbf{B}]_{+}$ For two square matrices A and B of the same size, $[A, B]_{+} = AB + BA$ is called their *anti-commutator*. When $[A, B]_{+} = 0$, we say that A and B anti-commute.
- $\delta(\mathbf{x}, \mathbf{y})$ Kronecker delta function. It equals 1 if $x = y$, and it equals 0 otherwise.
- $\mathbf{x} <= \mathbf{y}$ Same as $x \leq y$, x is less than or equal to y.
- $\mathbf{x} >= \mathbf{y}$ Same as $x \geq y$, x is greater than or equal to y.

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More Terminology

This section adds more definitions to the list given in the "Net Terminology" section of the Manual. The list of definitions given below is ordered so that each definition only uses terms that have been defined in previous definitions.

About our format:

Throughout the Quantum Fog documentation, we use parentheses in at least 2 different ways.

- We might say "...X1 (X2)..." to mean that the term or phrase X2 is a synonym or further explains the term or phrase X1, or
- we might say "...X1 (ditto, X2)...", to mean that everything that we are saying for X1 applies also to X2.

1. *internal (ditto, external) measurement (of a physical system being modelled by a net)* Physical realization of making inactive one or more (but not all) of the states of an internal (ditto, external) node X of the net. For example, a measurement might correspond to making inactive all but one of the states of node X. The measurement must be such that it sheds information about the state of X without disturbing the system so much that the net no longer models the system.

2. *coherence (interference)* Two stories are said to be coherent or to interfere if they have the same ending.

3. *destructive interference* A collection of stories is said to interfere destructively if its stories have the same ending, and the sum of their amplitudes is zero. In Quantum Mechanics, there are 2 cases in which a particular ending is unobservable. Either (1) there are no possible stories with that ending, or (2) there are several stories with that ending, but they interfere destructively. These two situations are not equivalent. For case 2, it is usually possible to make an internal measurement that prevents some but not all of the interfering stories from occurring. Such a measurement makes observable the previously unobservable ending. For case 1, there is no analogous mechanism for "resurrecting" an ending.

4. *correlated properties, correlated particles* For any random variable A, let $< A >$ be its mean value, and let $\Delta A = A - < A >$. Let each of N propertiesbe represented by a random variable. We say the$N$ random variables $X_1, X_2, ..., X_N$ and the properties they represent are correlated if and only if $< \Delta X_1 \Delta X_2 ... \Delta X_N >$ is non-zero. We say that N particles are correlated if and only if for each j such that $1 \leq j \leq N$, one can find a property X_j of particle j so that the properties $X_1, X_2, ..., X_N$ are correlated.

5. *spin* A particle property which characterizes the particle's symmetry under rotations. Quantum mechanical spin behaves much the same way as classical spin. In both classical and quantal physics, total angular momentum $\vec{J}$ equals orbital angular momentum $\vec{L}$ plus spin $\vec{S}$. $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ and $\vec{p}$ are the position and momentum vectors of the system. Spin is often called an intrinsic angular momentum, because it is independent of $\vec{r}$ and $\vec{p}$. The total angular momentum of an isolated system is conserved.

   One can measure the spin magnitude S of a particle. One can also measure its spin projection (or spin component) S_z along an arbitrary direction Z. For any given particle, its S can't be changed by any experiment that does not break the particle apart, but its S_z can be. S must be a non-negative integer or half-integer if measured in units of $\hbar$. S_z can only assume the values $S, S-1, S-2$, and so on, down to $-S$. For example, for an electron, a proton and a neutron, $S = 1/2$, and S_z must be either $1/2$ or $-1/2$. $S_z = -1/2$ is referred to as spin down or spin $-$, and $S_z = +1/2$ as spin up or spin $+$. Photons have $S = 1$, and their S_z must be either 1, 0 or -1.

   For spin 1/2 particles, we will often use the symbols u, $+$ and (0,1) as synonyms denoting $+$ spin component along a particular quantization direction. Likewise, d, $-$ and (1,0) will often be used as synonyms denoting $-$ spin component.

   Be careful: when we and others use the terms spin up and spin down, we are referring to the spin projection. When we say that a photon has spin 1, we are referring to the spin magnitude. Try not to get confused by this unfortunate practice of using the word "spin" to refer to both spin projection and spin magnitude. If you see the word "spin" being used without qualification as to whether the magnitude or the projection is meant, you'll have to figure out which one is meant from the context.

   Another Caveat: Experimentally, when we measure the spin magnitude of a particle, what we obtain is the value $\sqrt{S(S+1)}$ (i.e. we obtain the eigenvalue of $|\vec{S}_{op}|$, where $\vec{S}_{op}$ is the operator corresponding to the spin angular momentum vector). Thus, strictly speaking, we are incorrect in calling S the spin magnitude. But this is such a small peccadillo that we will continue to commit it unabashed. No one is fooled by it. After all, $\sqrt{S(S+1)}$ is an irrational number and S is an integer or half-integer, so it is easy to tell by the value which one is

meant.

6. *boson* Particle with integral spin magnitude (like photons, which have spin 1).

7. *fermion* Particle with half-integral spin magnitude (like protons, neutrons and electrons, which have spin 1/2).

8. *internal property* A particle property which can't change (as far as we know) during the course of the experiment under consideration. For example, the mass, charge and spin magnitude of a so called elementary particle (like the electron, proton, neutron or photon) can't change during the course of an experiment characterized by energies of a few electron volts. We are pretty sure about that.

9. *identical particles* Two particles are identical if there is no internal property for which they have different values.

10. *indistinguishable particles* Two identical particles (like, for example, two electrons), both of which lie in a state of definite position (ditto, momentum) are indistinguishable if and only if they have the same position (ditto, momentum). More generally, two identical particles are indistinguishable if and only if there is no measurement that can be performed which always selects the same particle. Note that two indistinguishable particles must be identical but not the converse. Indeed, two spatially separated and localized electrons are identical but distinguishable. In our discussions of Beam-Splitters, Polarization Rotators and Polarizers, whenever we refer to N photons, we mean N indistinguishable photons, N photons with the same polarization, same , and travelling in the same direction. The transition of a net from where 2 of its particles are distinguishable to where they are indistinguishable can be highly discontinuous. The number of possible stories might decrease, and several stories with different endings might be replaced by a single story. This highly discontinuous transition explains why the subject of identical particles is such a big deal in Quantum Mechanics.

11. *cbit* A physical system that behaves classically and can assume two states that we will call 0 and 1.

12. *qbit* (This word is often spelled "qubit", but we spell it "qbit" so we can also use the word "cbit".) A physical system that behaves quantically and has two basis states that we will call $|0\rangle$ and $|1\rangle$.

    In Quantum Mechanics, if a system has 2 basis states, it can occupy any "intermediate" quantum state $a|0\rangle + b|1\rangle$ where $a$and b are complex numbers. This intermediate quantum state is the "actual" state of the qbit; i.e., it is independent of the knowledge of any particular observer. By contrast, the actual state

of a cbit is always either 0 or 1. A cbit that behaves randomly is often characterized by a probabilistic "imagined" state that is intermediate between 0 and 1. This imagined state is not the actual state; i.e., it is observer dependent. It is a way of describing the imperfect knowledge of a particular observer.

We use the word qbit to describe any two-state quantum system. The system may be a particle (like an electron, proton or neutron) but it need not be. For example, it might consist of two energy levels of a complicated atom, in which case it isn't really a particle. Even if it isn't a particle, it can still be described by spin 1/2 formalism, but this is only an analogy. There really isn't a physical spin angular momentum associated with the system. The appendix entitled "Spin 1/2 Particles" discusses formalism more commonly used when we speak of qbits which are truly particles. The appendix entitled "Qbits" discusses formalism more commonly used when we speak of general qbits, without committing ourselves as to whether they are particles or not.

In the style of any good dictionary, we feel compelled to give some examples of the usage of the word qbit:

GEN 6:15 And this is the fashion which thou shalt make it of: The length of the ark shall be three hundred cubits, the breadth of it fifty cubits, and the height of it thirty cubits.

GEN 7:20 Fifteen cubits upward did the waters prevail; and the mountains were covered.

(i.e., when they build a quantum computer with more than 15 qbits, present day digital computers will be eclipsed.) ...And thou shall build a quantum computer with $300 \times 50 \times 30 = 450,000$ qbits and thou shall call it Noah's Ark.

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Some General References

This is a VERY short list of general references about Quantum Mechanics and measurement theory.

Quantum Mechanics Textbooks:

1. R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, Mass, 1963).

2. R.P. Feynman, A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGrawHill, 1965).

3. C. Cohen-Tannoudji, B. Diu, F. Laloe, *Quantum Mechanics*, 2 vols., (Hermann, Paris, 1977).

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Expanded Versus Summed Descriptions Of a Quantum Process

A quantum process and any Quantum Bayesian Net that represents it can be described in terms of state vectors (a "summed" description) or in terms of Feynman Paths (an "expanded" description). Let's explain what we mean with an example.

Open the Quantum Fog file entitled "SimpleQbitRot".

- STATES: Nodes *Root* and *Rot* both have 2 states, 0 and 1.
- AMPLITUDES: The amplitudes of *Root* are $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}\exp(i\theta)$ for states 0 and 1, respectively, where $\theta = 30^o$. The amplitudes of *Rot* were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info.** window. We set $\theta_2 = 45^o$ and all other angles to zero.

  If one represents $|0\rangle$ and $|1\rangle$ by

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \tag{1}$$

  then the amplitudes of *Rot* are given by the matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} . \tag{2}$$

  (U rotates a qbit by 90 degrees about the Y axis. See the appendix entitled "Qbits" for details.)
- INACTIVE STATES: All node states are active.

Expanded Description

The net "SimpleQbitRot" may be described in terms of the amplitudes of its stories as follows.

The net has 2 possible endings, $Rot = 0$ and $Rot = 1$. There are 2 stories with each of these 2 endings.

The amplitudes of the stories that end with $Rot = 0$ are:

$$A_{00} = \langle 0|U|0\rangle \frac{1}{\sqrt{2}} = \frac{1}{2} \,, \tag{3a}$$

$$A_{01} = \langle 0|U|1\rangle \frac{e^{i\theta}}{\sqrt{2}} = \frac{e^{i\theta}}{2} \,. \tag{3b}$$

Define FI_0 by

$$FI_0 = A_{00} + A_{01} = \frac{1 + e^{i\theta}}{2} \,. \tag{4}$$

The amplitudes of the stories that end with $Rot = 1$ are:

$$A_{10} = \langle 1|U|0\rangle \frac{1}{\sqrt{2}} = \frac{-1}{2} \,, \tag{5a}$$

$$A_{11} = \langle 1|U|1\rangle \frac{e^{i\theta}}{\sqrt{2}} = \frac{e^{i\theta}}{2} \,. \tag{5b}$$

Define FI_1 y

$$FI_1 = A_{10} + A_{11} = \frac{-1 + e^{i\theta}}{2} \,. \tag{6}$$

As expected, one gets

$$|FI_0|^2 + |FI_1|^2 = 1 \,. \tag{7}$$

Summed Description

The net "SimpleQbitRot" can also be described by giving the state vector at each stage of the system's evolution.

The initial state vector is

$$|\phi_{in}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\theta}|1\rangle) \,. \tag{8}$$

Using Eqs.(1) and (2) for $|0\rangle$, $|1\rangle$ and U, we see that the effect of U on $|\phi_{in}\rangle$ is

$$|\phi_{in}\rangle \rightarrow \left[\frac{1 + e^{i\theta}}{2}\right] |0\rangle + \left[\frac{-1 + e^{i\theta}}{2}\right] |1\rangle \,. \tag{9}$$

Comparison

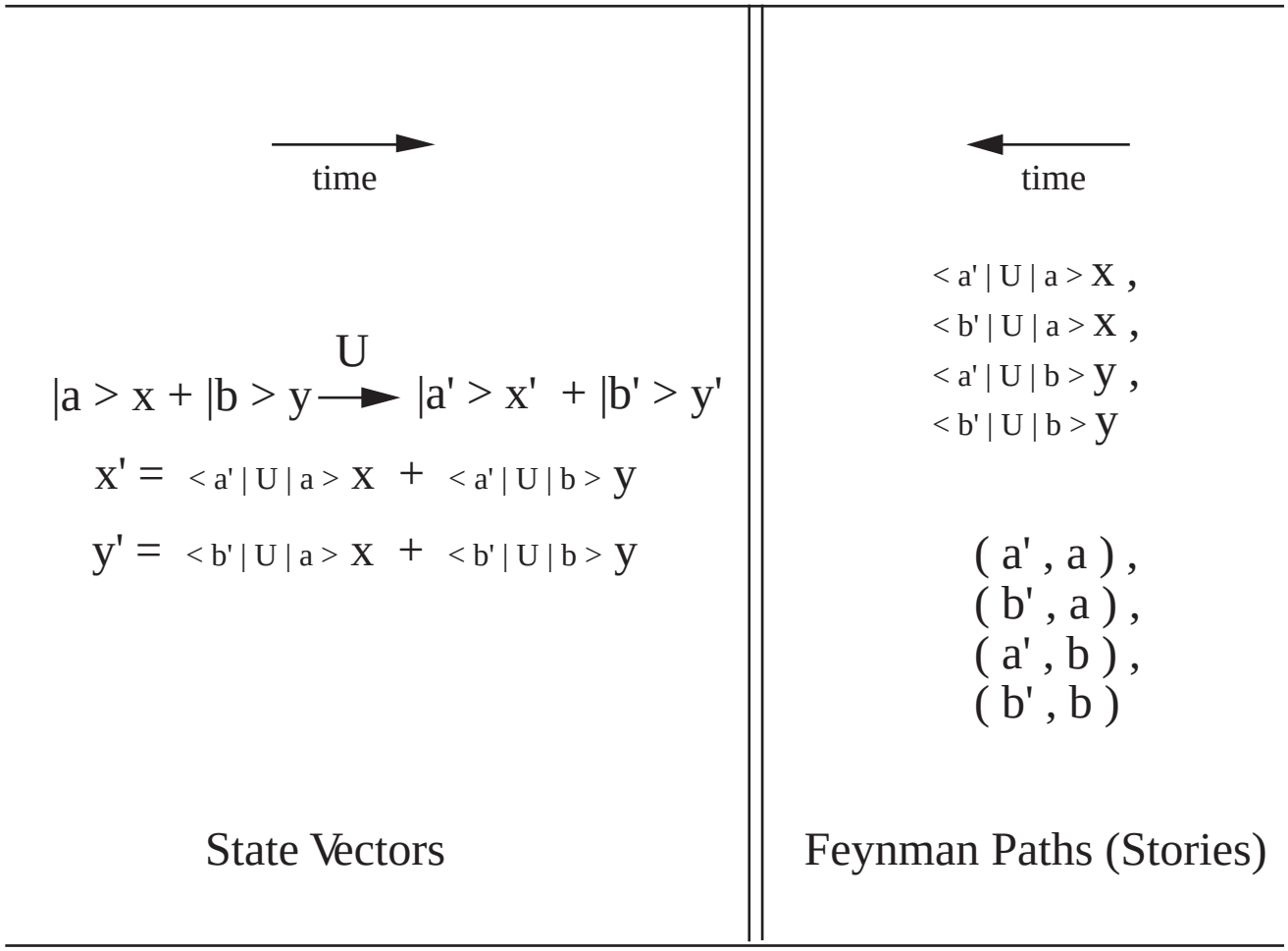

Fig. 1

Which description, either expanded or summed, is better depends on which Bayesian net and which property of the net is being considered. It is also a matter of personal taste.

I find that the expanded description is more visually intuitive than the summed one.

I find that the summed description is usually more compact. However, in processes with a large number of steps, the final coefficients of each eigenstate (for the example above, the coefficients $\frac{1}{2}(\pm 1 + \exp(i\theta))$ in Eq.(9)) can get quite complicated if they are expressed algebraically. On the other hand, if they are replaced by pure

numbers (for example, replacing $\frac{1}{2}(\pm 1 + \exp(i\theta))$ by 0 and 1 when $\theta = 0$), a lot of interesting patterns are lost. The expanded version is usually less compact than the summed one. It deals with more terms, one term for each story. However, the individual terms (for example, A_{00} in Eq.(3a)) are very simple. Furthermore, often many of the stories have zero amplitude. These terms can be ignored, which makes the expanded description more compact than might have seemed initially.

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Introduction To Toggling A Probability between $P(Ext)$ and $P(Ext|Int)$

It is possible to devise nets for which:

- a probability $P(Ext)$ equals zero, where Ext specifies the state of one or more external nodes of the net (For example, if $X1$ and $X2$ are external nodes of the net, then Ext might stand for the statement that $X1 = 1$ and $X2 = 0$.)
- by making a particular internal measurement Int, you can make the conditional probability of Ext, $P(Ext|Int)$, become non-zero.

With such nets, you can toggle the probability of Ext between a zero $P(Ext)$ and a non-zero $P(Ext|Int)$. The next few nets that we will discuss are examples of such nets.

Before embarking in the study of these nets, it is helpful to realize that this toggling of probabilities is related to the concepts of particle-wave duality, tracks and quantum erasers:

When $P(Ext)$ is zero, the system is said to exhibit wave-like behavior. That's because the zero $P(Ext)$ is due to destructive interference, and wave phenomena exhibit destructive interference whereas classical particles do not. When the probability $P(Ext|Int)$ is non-zero, the system is said to exhibit particle-like behavior. This observation that particle-like and wave-like behavior are mutually exclusive and yet may occur to the same system under different conditions is often called *wave-particle duality or complementarity*.

An internal measurement (for example, making inactive all but one state of an internal node X_{int} of a net) reduces the number of possible stories for the net. Thus, it increases our knowledge about the system's past, it *uncovers the system's tracks*. Conversely, making active all states of X_{int} decreases our knowledge about the system's past, it covers or *erases the system's tracks*.

If the system exhibits particle-like behavior, then it leaves tracks. If it exhibits wave-like behavior, then it covers its tracks.

Systems that can be made to erase or un-erase their tracks are sometimes called *quantum erasers*.

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Young's Double Slit Experiment

In Young's double slit experiment, light from a point source impinges on a planar surface. The surface is far from the source and has two closely spaced, parallel slits on it. The light which gets through the slits impinges on a screen. The screen is far from the slitted surface and parallel to it. Fringes, i.e. alternating dark and bright bands, show up on the final screen. If photographic emulsion is placed on the final screen, and the point source is made sufficiently weak, you can see the fringes form on the emulsion, one dot at a time. The simplest assumption is that the particle (photon) responsible for one dot on the emulsion has passed through one of the two slits. To check this, you place near one of the slits a device that, without blocking the photon's passage, detects whether the photon passes through that slit or not. To your amazement, you find that when you do so, the fringes go away. In particular, where before there was a dark band on the screen (zero probability of getting a photon) now there is a finite probability. In the language of Quantum Fog, you've managed to toggle the probability of a state of an external node by performing or not performing an internal measurement.

Open the Quantum Fog file entitled "Young'sDoubleSlitExp". We will use this net to analyze Young's experiment for a single photon.

- STATES: The node *Root* has two states: $left\_slit$ and $right\_slit$. The node NL (ditto, NR) has two states, 0 and 1. These integers represent the number of photons that pass through the left (ditto, right) slit. Suppose we give the name *Spot* to a fixed (i.e., same throughout the experiment) infinitesimal area of the screen. The node *Screen* has 2 states: *Spot* and *Other*. *Screen* equals *Spot* if the photon lands on *Spot*, whereas it equals *Other* if the photon lands elsewhere.

- AMPLITUDES: We define the amplitudes of *Root* to be $1/\sqrt{2}$ for both of its states. Let θ_L, θ_R, α_L, α_R be angles, and let f be a positive real number that is much smaller than 1. For the amplitudes of *Screen*, we use:

$A(Screen\|NL, NR)$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
Spot	0	$f\exp(i\theta_L)$	$f\exp(i\theta_R)$	0
Other	1	$(\sqrt{1-f^2})\exp(i\alpha_L)$	$(\sqrt{1-f^2})\exp(i\alpha_R)$	1

- INACTIVE STATES: Initially, all node states are active. Later on, you will be asked to make some states inactive.

The net has 2 possible endings, one for each state of $Screen$. There are two possible stories for each ending, depending on whether the particle goes through the left or the right slit. The amplitudes for the two stories ending in $Screen = Spot$ are

$$A_{L,Spot} = f e^{i\theta_L} \frac{1}{\sqrt{2}} \ , \ \ A_{R,Spot} = f e^{i\theta_R} \frac{1}{\sqrt{2}} \ . \tag{1}$$

Define the Feynman Integral FI_{Spot} by

$$FI_{Spot} = A_{L,Spot} + A_{R,Spot} \ . \tag{2}$$

The amplitudes for the two stories ending in $Screen = Other$ are

$$A_{L,Other} = \sqrt{1-f^2} \ \ e^{i\alpha_L} \frac{1}{\sqrt{2}} \ \ , \ \ A_{R,Other} = \sqrt{1-f^2} \ \ e^{i\alpha_R} \frac{1}{\sqrt{2}} \ . \tag{3}$$

Define the Feynman Integral FI_{Other} by

$$FI_{Other} = A_{L,Other} + A_{R,Other} \ . \tag{4}$$

We require that

$$|FI_{Spot}|^2 + |FI_{Other}|^2 = 1 \ . \tag{5}$$

Plugging Eqs.(1) to (4) into Eq.(5) yields the constraint:

$$\cos(\alpha_L - \alpha_R) = \frac{-f^2 \cos(\theta_L - \theta_R)}{1 - f^2} \ . \tag{6}$$

In the Quantum Fog file entitled "Young'sDoubleSlitExp", we've used $\theta_R = 0$, $\alpha_R = 0$, $\theta_L = 180^0$, and $f = 0.1$. To satisfy Eq.(6), we also used $\alpha_L = 89.421^0$. You can verify that the net "Young'sDoubleSlitExp" does indeed satisfy Eq.(5) by pressing the **Preview** item in the **Run_Preparations** menu.

If all node states are active, the probability that $Screen$ equals $Spot$ is zero.

$$P(Screen = Spot) = |A_{L,Spot} + A_{R,Spot}|^2 = 0 \ . \tag{7}$$

If you now make inactive all states of NL except 1, the probability that $Screen$ equals $Spot$ becomes non-zero.

$$P(Screen = Spot | NL = 1) = \frac{|A_{L,Spot}|^2}{|A_{L,Spot}|^2 + |A_{L,Other}|^2} = f^2 = 0.01 \ . \tag{8}$$

What-ho? What-ho? By performing or not performing an internal measurement (the one which makes $NL = 1$), you've managed to toggle (between the values of 0 and 0.01) the probability that the external node $Screen$ equals $Spot$.

Why is the change in probability so small (just 0.01)? Because *Spot* is very small compared to *Other*, so most of the time the photon lands on *Other*. If *Screen* had more states, like a *Spot*1 and *Spot*2 of equal area, then we could divide the probability that the photon landed on *Spot*1 by the probability that it landed on *Spot*2. This ratio would no longer be proportional to the infinitesimal size of spots, and, therefore, could be much larger than 1%.

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Humpty Dumpty

This experiment is analogous to Young's Double Slit Experiment, except that it uses a fermion instead of a boson. The slitted surface is replaced by a Stern-Gerlach magnet and so is the screen. As in Young's experiment, by performing or not performing a particular internal measurement we can toggle the probability of a state of an external node.

Apart from being a net that allows the toggling of a probability of an external node, this net is remarkable in that it breaks up a quantum state, and then puts it back together again. The net reverses disorder, in apparent contradiction to the Second Law of Thermodynamics. What's even more shocking, ladies and gentlemen of the jury, is that this net shows conclusively, beyond a shadow of a doubt, that Humpty Dumpty's life could have been spared. (Gasp!)

Your honor, I would like at this moment to present exhibit A, the Quantum Fog file entitled "HumptyDumpty".

- STATES: Node *root* has a single state (0,1), indicating that a single particle, with spin + in the X direction, always enters the device. The Stern-Gerlach magnet *z_mag* has 2 states : (1,0) and (0,1). The Stern-Gerlach magnet *x_mag* has 3 states : (0,0), (1,0) and (0,1). For the states of both magnets, the first integer indicates the number of particles with spin − and the second the number with spin +. All the deterministic nodes ($nz+, nz-, nx+$ and $nx-$) have two states: 0 and 1. These integers are the number of particles passing through the node.

- AMPLITUDES: The node *root* is in state (0,1) with unit amplitude.

  The amplitudes of node *z_mag* were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info.** window. We used as the quantization direction of input node *root* the positive X direction, and as the output quantization direction the positive Z direction.

  The amplitudes of node *x_mag* were also obtained by pressing the **Generate Amplitudes...** button. We used as the quantization direction of input nodes $nz+$ and $nz-$ the positive Z direction, and as the output quantization direction the positive X direction.

- INACTIVE STATES: Initially, all node states are active. Later on, you will be asked to make some states inactive.

This net has two possible endings, depending on whether the particle ends at the $nx+$ or the $nx-$ node. Each ending has two possible stories. According to the appendix entitled "Stern-Gerlach Magnet", the amplitudes of these stories are as follows. The amplitudes for the stories that end in $nx+ = 1$ are:

$$A_{t\to t} = \langle +_x|+_z\rangle\langle +_z|+_x\rangle = \frac{1}{2}\ , \tag{1a}$$

$$A_{b\to t} = \langle +_x|-_z\rangle\langle -_z|+_x\rangle = \frac{1}{2}\ . \tag{1b}$$

(Here b stands for bottom and t for top.) The amplitudes for the stories that end in $nx- = 1$ are:

$$A_{t\to b} = \langle -_x|+_z\rangle\langle +_z|+_x\rangle = \frac{-1}{2}\ , \tag{2a}$$

$$A_{b\to b} = \langle -_x|-_z\rangle\langle -_z|+_x\rangle = \frac{1}{2}\ . \tag{2b}$$

With all node states active, the particle never ends up at $nx-$:

$$P(nx- = 1) = |A_{t\to b} + A_{b\to b}|^2 = 0\ . \tag{3}$$

However, if we make inactive all states of node $nz+$ except 1, then the particle has a 50% chance of ending at $nx-$:

$$P(nx- = 1|nz+ = 1) = \frac{|A_{t\to b}|^2}{|A_{t\to t}|^2 + |A_{t\to b}|^2} = \frac{1}{2}\ . \tag{4}$$

Thus, this is an example of toggling a probability of an external node by performing or not performing an internal measurement.

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Toggling A Bi-node Probability

This net represents an experiment that was first discussed in Refs.[1] and [2]. By performing or not performing a particular internal measurement on the net, we will toggle the probability of a state of an external bi-node. The bi-node probability to be toggled is a coincidence probability (i.e., the probability that 2 separate detectors each detects a particle.)

Open the Quantum Fog file entitled "TogglingABiNodeProb".

- STATES: Nodes $in1$ and $in2$ both have a single state (1,0), which represents a single photon polarized in the X direction. Node *rot*, a polarization rotator, has a single state (0,1), which represents a single photon polarized in the Y direction. Node *bs* has states of the form appropriate for a beam-splitter with vector-field inputs. If (N_1, N_2) is the state of node *bs*, where N_1 and N_2 is each a 2-component vector, then the states of e_field1 and e_field2 are N_1 and N_2, respectively. Nodes $pol1$ and $pol2$ have states of the form appropriate for polarizers. The states of $pol1\_copy$ (ditto, $pol2\_copy$) are identical to those of $pol1$ (ditto, $pol2$). Node $out1$ (ditto, $out2$) has 3 states: 0, 1 and 2, which represent the number of particles that pass through $pol1$ (ditto, $pol2$) without being absorbed.

- AMPLITUDES: Nodes $in1$ and $in2$ both are assigned unit amplitude of being entered by a single X polarized photon. The amplitudes of nodes $rot, bs, pol1$ and $pol2$ were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info.** window. For rot, we used $\theta = 90^o$ as the rotation angle. For bs, we used $|t|^2 = \frac{1}{2}$, phase$(t) = 0$ and phase$(r) = 90^o$. For $pol1$ and $pol2$, we used $\theta = 45^o$ as the angle between the X axis and the polarization direction.The amplitudes of the deterministic nodes $e\_field1, e\_field2, out1$ and $out2$ are self-explanatory. Note that the transition matrices of nodes $pol1\_copy$ and $pol2\_copy$ are just identity matrices. Later on, we'll explain why these nodes are necessary.

- INACTIVE STATES: Initially, all node states are active. Later on, you will be asked to make some states inactive.

Why are the "copy" nodes $pol1\_copy$ and $pol2\_copy$ necessary? Let $\mathcal{N}$ be the net given, and let $\mathcal{N}_r$ be the reduced net obtained by removing nodes $pol1\_copy$ and $pol2\_copy$ from $\mathcal{N}$. $\mathcal{N}_r$ yields a "probability" of 1.7 when you press the **Preview**

item of the **Run_Preparations** menu. Running $\mathcal{N}_r$ with all states active yields 26 stories with 6 different endings. In contrast, $\mathcal{N}$ yields a probability of 1 when you press **Preview**. Running $\mathcal{N}$ with all states active yields 26 stories with 17 different endings. The 26 stories of $\mathcal{N}$ are in 1-1 correspondence with the 26 stories of $\mathcal{N}_r$, where corresponding stories assign the same state to those nodes which are common to both nets. Certain stories that interfere in $\mathcal{N}_r$ have corresponding stories in $\mathcal{N}$ which do not interfere. Thus, we see that the effect of including the copy nodes is to remove certain undesired interferences between stories.

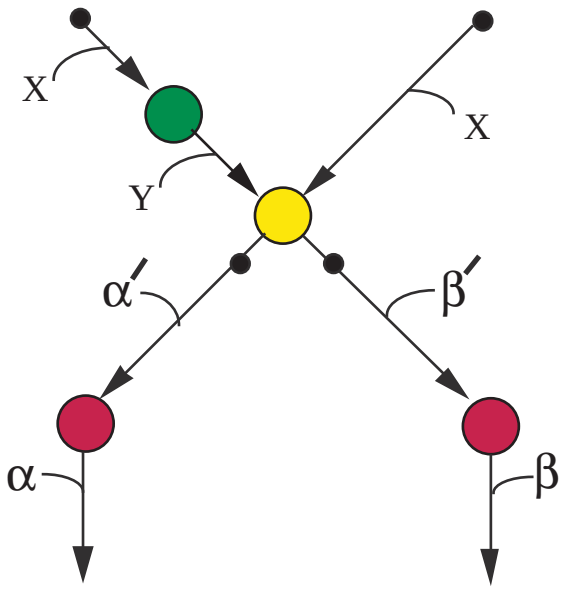

FIG. 1

Next we will calculate the probability $P(out1 = 1, out2 = 1)$ (coincidence probability) for the bi-node $(out1, out2)$. This probability can be expressed in terms of the story amplitudes $A^{\alpha',\beta'}_{\alpha,\beta}$, where α', β', α and β belong to the set $\{X, Y\}$. Here X and Y stand for polarization along the X and Y directions. See Fig.1 for the story with amplitude $A^{\alpha',\beta'}_{\alpha,\beta}$. This figure gives the direction of polarization for the particle that flows through each arrow. (The polarization direction defines the state of the arrow. The state of an arrow is the same as the state of its source node). Define

$$t = \frac{1}{\sqrt{2}} \quad , \quad r = \frac{i}{\sqrt{2}} \, , \tag{1a}$$

$$C = \cos(45^o) = \frac{1}{\sqrt{2}} \quad , \quad S = \sin(45^o) = \frac{1}{\sqrt{2}} \, . \tag{1b}$$

From the appendices entitled "Beam-Splitter" and "Polarizer", it follows that

$$A^{Y,X}_{\alpha,\beta} = (-|r|^2)[CS\delta(\alpha, X) + S^2\delta(\alpha, Y)][C^2\delta(\beta, X) + CS\delta(\beta, Y)] = \frac{-1}{8} \, , \tag{2a}$$

$$A^{X,Y}_{\alpha,\beta} = (|t|^2)[C^2\delta(\alpha,X) + CS\delta(\alpha,Y)][CS\delta(\beta,X) + S^2\delta(\beta,Y)] = \frac{1}{8}\ . \qquad (2b)$$

Therefore,

$$P(out1 = 1, out2 = 1) = k\sum_{\alpha}\sum_{\beta}|A^{Y,X}_{\alpha,\beta} + A^{X,Y}_{\alpha,\beta}|^2 = 0\ . \qquad (3)$$

Here k is a normalization constant, and the sums over α and β both range over the set $\{X, Y\}$.

If you make inactive all states of node e_field1 except for state (1,0), then

$$P[out1 = 1, out2 = 1|e\_field1 = (1,0)] = k'\sum_{\alpha}\sum_{\beta}|A^{X,Y}_{\alpha,\beta}|^2 \neq 0\ . \qquad (4)$$

Thus we see that it is possible to toggle the probability of a state of an external bi-node by performing or not performing an internal measurement.

References

[1] Y.H. Shih, C.O. Alley, Phys.Rev. Let. **61**, 2921 (1987).

[2] P.G. Kwait, A.M. Steinberg, R.Y. Chiao, Phys. Rev. A **45**, 7729 (1992).

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Watched Pot Never Boils

Next we will study an effect that belongs to a class of effects referred to in the physics literature by the maxim "A watched pot never boils" or by the term "Zeno's Paradox". (The paradox being alluded to is one of several paradoxes propounded by the Greek philosopher Zeno in the 4th century B.C. It goes as follows: a flying arrow, when viewed at a fixed moment in time, appears stationary.) The common unifying trait of all watched pot effects is that they deal with a physical system that experiences small but frequent outside influences which incline it not to stray far away from its initial state. Watched pot effects can occur in purely classical systems or in quantum ones. See Refs.[1]-[2].

One particular watched pot effect that occurs in quantum systems is the following. Suppose a system, known to occupy at time t a quantum state Ψ_1, is left alone over the time interval $[t, t+\Delta t]$, and is then measured at time $t+\Delta t$. For such a system, we call $P_{1\leftarrow 1}(\Delta t)$ the probability that at time $t+\Delta t$, the system will be found to be still lying in state Ψ_1. Suppose that $P_{1\leftarrow 1}(\Delta t) \rightarrow 1$ continuously as $\Delta t \rightarrow 0$ and that τ is a positive real number that is small enough for $P_{1\leftarrow 1}(\tau)$ to be fairly close to 1. Suppose that the system is measured at times $t = \frac{i}{N}\tau$ for $i = 1, 2, \ldots, N$, where N is some positive integer. The probability that the system will be found to be in state Ψ_1 in every one of those N measurements is then $[P_{1\leftarrow 1}(\frac{\tau}{N})]^N$. As we show in the proof below, this probability tends to 1 as N tends to infinity. Thus, each measurement exerts a small influence on the system. These influences incline the system not to stray far away from its initial state Ψ_1. It's not clear who was the first person to discover this effect. The authors of Ref.[3] were the first to call it "Quantum Zeno's Paradox". The effect has been observed experimentally[4] in atoms executing induced transitions between energy levels.

Proof: If H is the Hamiltonian of the system,

$$P_{1\leftarrow 1}(\Delta t) = |\langle \Psi_1| \exp(-i\frac{H\Delta t}{\hbar})|\Psi_1\rangle|^2 \approx 1 - (\delta\omega)^2(\Delta t)^2 + \cdots \ , \tag{1}$$

where

$$\hbar\delta\omega = \sqrt{\langle\Psi_1|H^2|\Psi_1\rangle - [\langle\Psi_1|H|\Psi_1\rangle]^2} \ . \tag{2}$$

Hence, as $N \rightarrow \infty$

$$[P_{1\leftarrow 1}(\frac{\tau}{N})]^N \approx [1 - (\delta\omega)^2 \left[\frac{\tau}{N}\right]^2]^N \approx \exp\left[-(\delta\omega)^2\frac{\tau^2}{N}\right] \rightarrow 1 \ . \tag{3}$$

You might ask: Since watched pot effects are not directly related to toggling a probability, why is this watched pot stuff being discussed in the toggling a probability section? Well... This watched pot stuff is intended as a prologue to the section entitled "Enhanced Interaction-free Detection", which combines probability toggling and a watched pot effect. Capiche?

Open the Quantum Fog file entitled "WatchedPot". This net will be used to illustrate a watched pot effect that was first discussed in Refs[2] and [5]—an effect which, although different from the atomic watched pot effect mentioned above, is very similar to it.

- STATES: As usual, we will say that a state is a vector-field state if it is of the form (N_x, N_y), where N_x and N_y are non-negative integers that represent the number of photons polarized in the X and Y directions. The deterministic nodes labelled $in1, in2, \cdots, in6$, and out all have 2 vector-field states: $(0,0)$ and $(1,0)$. All polarization rotator nodes have 3 vector-field states: $(0,0)$, $(1,0)$ and $(0,1)$.

  All polarizer nodes have states $(0,0)0$, $(1,0)0$ and $(0,0)1$. Here the first and second integers are the number of photons polarized in the X and Y directions. The last integer is the number of photons that are absorbed by the polarizer.

  The deterministic nodes labelled $loss1$ up to $loss6$ have states 0 and 1. These state names represent the number of photons that reach the loss node.

- AMPLITUDES: The amplitudes assigned to the deterministic nodes are self-explanatory. Note that node $in1$, which is the only root node, is in state $(1,0)$ with unit amplitude. Hence, a single photon polarized in the X direction enters the single input port on the left side.

  The amplitudes of all the polarization rotators and polarizers were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info.** window, using $\theta = 15^o$ for the polarization rotators and $\theta = 0$ for the polarizers.

- INACTIVE STATES: All node states are active.

The single photon that enters node $in1$ may end up in one of the seven nodes: $loss1, loss2, \cdots, loss6, out$. That is, the photon may either be absorbed by one of the 6 polarizers or it may emerge unscathed from the last polarizer. Hence, the net has 7 possible endings.

There is just one possible story with each of these 7 endings. According to the appendices entitled "Polarization Rotator" and "Polarizer", the amplitudes of these 7 stories are:

$$A_j = C^{j-1}S \quad \text{for } \; j = 1, 2, \ldots, 6 \; , \tag{4a}$$

and

$$A_7 = C^6 \; , \tag{4b}$$

where

$$C = \cos(15^o), \quad S = \sin(15^o) \; . \tag{5}$$

The C and S factors in these amplitudes all come from the polarization rotators. All other nodes contribute factors of 1 to the story amplitudes. It's easy to show that

$$\sum_{j=1}^{7} |A_j|^2 = S^2[1 + C^2 + C^4 + \cdots + C^{10}] + C^{12} = 1 \; . \tag{6}$$

The probability that the photon is not absorbed by any of the polarizers is $|A_7|^2 = C^{12} = 0.660 \approx \frac{2}{3}$. The state entering $rot1$ is polarized along the X direction. There are 6 polarization rotators, and each of them rotates the polarization of the state entering it by 15 degrees. If the polarizers were removed, then the state leaving $rot6$ would be polarized along the Y direction, and an X polarized photon would never reach the *out* node. With the polarizers in place, however, an X polarized photon reaches the *out* node almost 2/3 of the time. If the net had N rotator-polarizer pairs instead of just six, and if the rotators had a rotation angle of $\frac{90^o}{N}$, then the probability that the photon would not be absorbed by any of the polarizers would be $\cos^{2N}(\frac{90^o}{N})$. This probability tends to 1 as N tends to infinity. The polarizers act as measuring devices. They monitor the evolution produced by the polarization rotators. This weakly intrusive monitoring inclines the photon not to stray far away from its initial X polarized state.

References

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[4] W.M. Itano, D.J. Heinzen, J.J. Bollinger, D.J. Wineland Phys. Rev. A **41**, 2295 (1990).

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Basic Interaction-free Detection

Suppose that in Young's double slit experiment, you placed a detector on the screen at the position of a dark fringe. Then suppose someone inserted, without telling you, an object that blocked one of the two slits. If the detector were to detect a photon, then you would immediately know 2 things: (1) that an obstruction to one of the slits existed, and (2) that the obstruction had not interacted with the photon. (1) and (2) would follow by contradiction. If not (1), then the detector could not have detected a photon because of destructive interference. If not (2), then the obstruction would have absorbed the photon, and once again, the detector could not have detected it. Of course, if the detector did not register a photon, then you couldn't say with certainty whether or not someone had inserted an obstruction.

This experiment suggests a way of detecting the presence of an obstruction within a region **R** of space. When asked whether there is an obstruction, the detector gives an answer of Yes a fraction ϵ of the time, and an answer of Uncertain $1-\epsilon$ of the time. We'll call ϵ the efficiency. The fraction ϵ of the time that it says Yes, no signal related to the experiment, whether outgoing or incoming, crosses the surface of **R**–the detection is *interaction-free*. (The other $1-\epsilon$ of the time, a signal may or may not cross the surface of **R**.)

Of course, ϵ would be very low in the double slit experiment just discussed. In this section, we will discuss a much more practical experiment, first discussed in Ref.[1], which gives interaction-free detection $\frac{1}{4}$ of the time. In a later section, we will discuss an experiment that gives interaction-free detection nearly all the time!

Interaction-free detection is simply a practical application of something that is already very familiar to us: toggling a probability of an external node by making an internal measurement. In the case of the Young's double slit experiment just discussed, the probability being toggled is the probability that the screen detector measures a photon, and the internal measurement being performed is photon absorption by the obstruction.

Open the two Quantum Fog files entitled "BasicIntFreeDetec1" and "BasicIntFreeDetec2". We will henceforth refer to these files as "Net1" and "Net2", respectively. Net1 is a Mach-Zehnder interferometer with a single photon as input. Net2 is the same interferometer, but with an obstruction in the right path.

Let's consider Net1 first:

- STATES: All deterministic nodes (i.e., all nodes except $BS1$ and $BS2$) have exactly 2 states: 0 and 1. The name of these states indicates the number of particles that pass through the node during the course of the experiment.

The beam-splitter nodes $BS1$ and $BS2$ have 3 states: $(0,0),(1,0),(0,1)$. The first (ditto, second) integer is the number of photons that exit the beam-splitter on the left(ditto, right) side.

- AMPLITUDES: The amplitudes assigned to the deterministic nodes are self-explanatory. Note that $i_L = 1$ and $i_R = 0$ both occur with unit amplitude. Thus, the Mach-Zehnder is entered by one photon at the top-left corner.

  The amplitudes of $BS1$ and $BS2$ were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info.** window, using

$$t_1 = t_2 = \frac{1}{\sqrt{2}} \ , \ \ r_1 = r_2 = \frac{i}{\sqrt{2}} \ . \tag{1}$$

- INACTIVE STATES: All node states are active.

Net1 has 2 possible endings: $(f_L, f_R) = (1,0)$ and $(0,1)$.

There are two possible stories with ending $(f_L, f_R) = (1,0)$. According to the appendix entitled "Beam-Splitter", the amplitudes of these stories are:

$$A(\sigma_1) = t_2 t_1^* \ , \tag{2a}$$

and

$$A(\sigma_2) = r_2 r_1 \ . \tag{2b}$$

Define $FI(1,0)$ to be the following Feynman Integral:

$$FI(1,0) = A(\sigma_1) + A(\sigma_2) \ . \tag{3}$$

There are two possible stories with ending $(f_L, f_R) = (0,1)$. According to the appendix entitled "Beam-Splitter", the amplitudes of these stories are:

$$A(\sigma_3) = (-r_2^*) t_1^* \ , \tag{4a}$$

and

$$A(\sigma_4) = (t_2^*) r_1 \ . \tag{4b}$$

Define $FI(0,1)$ to be the following Feynman Integral:

$$FI(0,1) = A(\sigma_3) + A(\sigma_4) \ . \tag{5}$$

The probability that $f_L = 1$ is:

$$P(f_L = 1) = \frac{1}{k} |FI(1,0)|^2 \ , \tag{6a}$$

where

$$k = |FI(1,0)|^2 + |FI(0,1)|^2 \ . \tag{6b}$$

Plugging the t_1, r_1, t_2, r_2 values given by Eq.(1) into Eqs.(2) to (6) yields

$$P(f_L = 1) = 0 \ . \tag{7}$$

Hence, f_L (i.e, the final, left detector) is always "dark", and f_R is always "bright". The photon always exits the device at the bottom-right corner.

Now consider Net2.

Net2 is identical to Net1 in every respect except that node d_R has been replaced by two nodes: *obstruction*, and *dark*. *obstruction* has 2 states: 0 and 1. These integers are the number of particles that arrive at the node. *dark* has only one state, 0; as the node's name implies, no particle ever passes through this node.

Net2 has 3 possible endings: $(f_L, f_R, obstruction) = (1,0,0), (0,1,0)$ and $(0,0,1)$. There is only one possible story with each of these endings. The amplitudes of these stories are:

$$FI(1,0,0) = r_2 r_1 \ , \tag{8a}$$

$$FI(0,1,0) = t_2^* r_1 \ , \tag{8b}$$

$$FI(0,0,1) = t_1^* \ . \tag{8c}$$

The probability that $f_L = 1$ is:

$$P(f_L = 1) = \frac{1}{k}|FI(1,0,0)|^2 \ , \tag{9a}$$

where

$$k = |FI(1,0,0)|^2 + |FI(0,1,0)|^2 + |FI(0,0,1)|^2 \ . \tag{9b}$$

Plugging the t_1, r_1, t_2, r_2 values given by Eq.(1) into Eqs.(8) to (9) yields

$$P(f_L = 1) = \frac{1}{4} \ . \tag{10}$$

Thus we see that the photon will arrive at detector f_L 25% of the time. Thus, 25% of the time, we get interaction-free detection of obstructions located in the right side of Net2.

Note that you cannot model the insertion of an obstruction into the right path of Net1 by simply making inactive all states of node d_R except $d_R = 0$. Such a net does not have a story in which the photon ends up being absorbed by the obstruction.

At the beginning of this section, I claimed that this experiment was just one more example of toggling a probability of an external node by performing or not performing an internal measurement. Well... I lied. In this experiment, including the obstruction is not really an internal measurement. As we've defined it, an internal

measurement refers to just one net. When we included the obstruction, the net representing the experiment changed from Net1 to Net2. The node d_R split into two nodes: *obstruction* and *dark*. It's as if we had set out to make an internal measurement on Net1, but our measurement was too violent, and it changed the nature of the net we started with. In some sense, inserting the obstruction is the limit of an internal measurement. Thus, we could say that the experiment of this section illustrates toggling the probability of a state of an external node by performing or not performing the "limit" of an internal measurement.

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Enhanced Interaction-free Detection

This section combines the results of the sections entitled "Watched Pot Never Boils" and "Basic Interaction-free Detection". It discusses an experiment, first discussed in Refs.[1], that allows the detection of an object in a way that is interaction-free nearly all the time.

Open the two Quantum Fog files entitled "EnhancedIntFreeDetec1" and "EnhancedIntFreeDetec2". We will henceforth refer to these files as Net1 and Net2, respectively.

Let's consider Net1 first. Let N stand for 6, the number of beam-splitters.

- STATES: All deterministic nodes (i.e., all nodes except BSj for $1 \leq j \leq N$) have exactly 2 states: 0 and 1. The name of these states indicates the number of photons that pass through the node during the course of the experiment.

  The beam-splitter nodes BSj, where $1 \leq j \leq N$, have 3 states: $(0,0), (1,0), (0,1)$. The first (ditto, second) integer is the number of photons that exit the beam-splitter on the left(ditto, right) side.

- AMPLITUDES: The amplitudes assigned to the deterministic nodes are self-explanatory. Note that $i_L = 1$ and $i_R = 0$ both occur with unit amplitude. Thus, the device is entered by one photon at the top-left corner.

  The amplitudes of nodes BSj for $1 \leq j \leq N$ were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info.** window, using

$$t_j = \sin(\frac{\pi}{2N}) \ , \ \ r_j = i\cos(\frac{\pi}{2N}) \ , \ \text{ for } 1 \leq j \leq N \ . \tag{1}$$

- INACTIVE STATES: All node states are active.

Net1 has 2 possible endings: $(f_L, f_R) = (1,0)$ and $(0,1)$. There are $2^{N-1} = 32$ possible stories with each ending. We want to calculate the probability $P(f_L = 1)$ that $f_L = 1$. This probability can be expressed in terms of the sums $FI(f_L = 1)$ and $FI(f_R = 1)$ of all the stories with the ending $(1,0)$ and $(0,1)$, respectively. Indeed,

$$P(f_L = 1) = \frac{1}{k}|FI(f_L = 1)|^2 \ , \tag{2a}$$

where

$$k = |FI(f_L = 1)|^2 + |FI(f_R = 1)|^2 \ . \tag{2b}$$

Recall the Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ , \tag{3a}$$

and the identity

$$e^{i\theta\sigma_x} = \cos\theta + i\sigma_x \sin\theta \ . \tag{3b}$$

Define U_j by

$$U_j = \begin{bmatrix} t_j & r_j \\ -r_j^* & t_j^* \end{bmatrix} \ , \ \text{for } 1 \le j \le N \ . \tag{4}$$

It's easy to check that if Net1 had one beam-splitter instead of N, then

$$\begin{bmatrix} FI(f_R = 1) \\ FI(f_L = 1) \end{bmatrix} = \sigma_x U_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ . \tag{5}$$

Define U by

$$U = \prod_{j=1}^{N} (\sigma_x U_j) \ . \tag{6}$$

When Net1 has N beam-splitters

$$\begin{bmatrix} FI(f_R = 1) \\ FI(f_L = 1) \end{bmatrix} = U \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ . \tag{7}$$

Using the values for t_j and r_j given by Eq.(1), one gets

$$\sigma_x U_j = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S & iC \\ iC & S \end{bmatrix} = \begin{bmatrix} iC & S \\ S & iC \end{bmatrix} = i \exp(-i\frac{\pi}{2N}\sigma_x) \ , \tag{8}$$

where $C = \cos(\frac{\pi}{2N}), S = \sin(\frac{\pi}{2N})$. From Eqs.(6) and (8) it follows that

$$U = i^{N-1}\sigma_x \ . \tag{9}$$

Plugging this value for U into Eq.(7) yields

$$\begin{bmatrix} FI(f_R = 1) \\ FI(f_L = 1) \end{bmatrix} = \begin{bmatrix} i^{N-1} \\ 0 \end{bmatrix} \ . \tag{10}$$

Thus, by virtue of Eqs.(2),

$$P(f_L = 0) = 0 \ . \tag{11}$$

Hence, f_L (i.e, the final, left detector) is always "dark", and f_R is always "bright". The photon always exits the device at the bottom-right corner.

Now consider Net2.

Net2 is identical to Net1 in every respect except that each node d_{Rj}, where $1 \leq j \leq N-1$, has been replaced by two nodes: *obstj*, and *darkj*. *obstj* has 2 states: 0 and 1. These integers are the number of photons that arrive at the node. *darkj* has only one state, 0; as the node's name implies, no photon ever passes through this node. For now, you can think of the nodes *obstj* as representing $N-1=5$ different obstructions (or else one very long one). Later we will show that the graph of Net2 can be "folded onto itself". In the folded graph, a single beam-splitter and a single obstruction act repeatedly, and perform the same function as multiple beam-splitters and multiple obstructions.

Net2 has $N+1=7$ possible endings, depending on whether the photon ends at f_L, f_R, or *obstj* for $1 \leq j \leq N-1$. There is only one possible story with each of these endings. As in the section entitled "Watched Pot never Boils", if the amplitudes for these stories are called A_j for $1 \leq j \leq 7$, then one can show that $\sum_{j=1}^{7} |A_j|^2 = 1$. $P(f_L = 1)$ equals the magnitude squared of one of those 7 amplitudes:

$$P(f_L = 1) = \left| \prod_{j=1}^{N} r_j \right|^2 = \cos^{2N} \frac{\pi}{2N} \; . \tag{12}$$

For $N = 6$, this last probability equals about 66%. Thus we see that the photon will arrive at detector f_L 66% of the time. Thus, 66% of the time, we get interaction-free detection of obstructions located in the right side of Net2.

Note that as N tends to infinity, the right side of Eq.(12) tends to 1. But note also that as N tends to infinity, Eq.(1) gives $t = 0$ and $r = i$ for all the beam-splitters, so the photon can never visit the right side of either Net1 or Net2. But that would imply that $P(f_L = 1) = 1$ for Net1. But according to Eq.(11), $P(f_L = 1) = 0$ for Net1. Thus, our model predicts two different answers for the probability $P(f_L = 1)$ of Net1 when N is infinite. The limit depends on the order in which we take it. So our model in not too trusty for extremely large N. This is OK, because the model must be invalid for extremely large N for other reasons as well. The model ignores physically unavoidable losses in the mirrors and beam-splitters. The experiment becomes more and more sensitive to these as N grows.

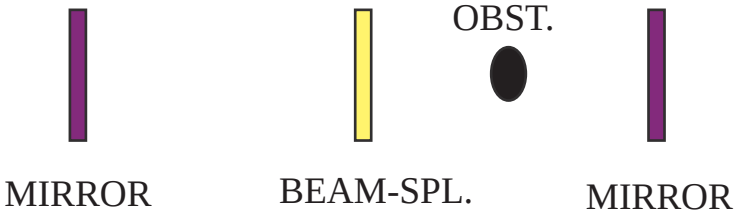

FIG. 1

Fig.1 shows how Net1 and Net2 can be folded so that only one beam-splitter and one small obstruction are required. A beam-splitter with reflection and transmission coefficients given by Eq.(1) is placed midway between two perfect mirrors. The obstruction, if there is one, is placed between the beam-splitter and the right mirror. The left mirror is momentarily removed to allow a photon to enter on the left side. After a time equal to NL/c, where L is the distance between the end mirrors and c is the speed of light, both mirrors are removed to allow the photon to exit.

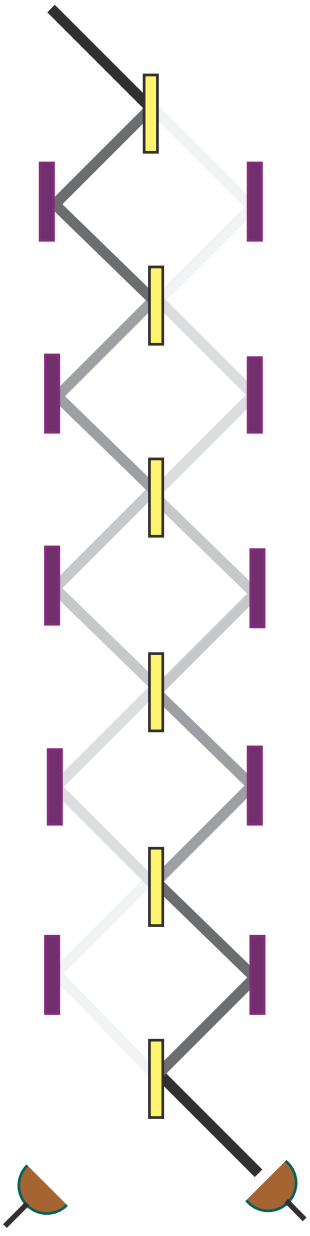

FIG.2a

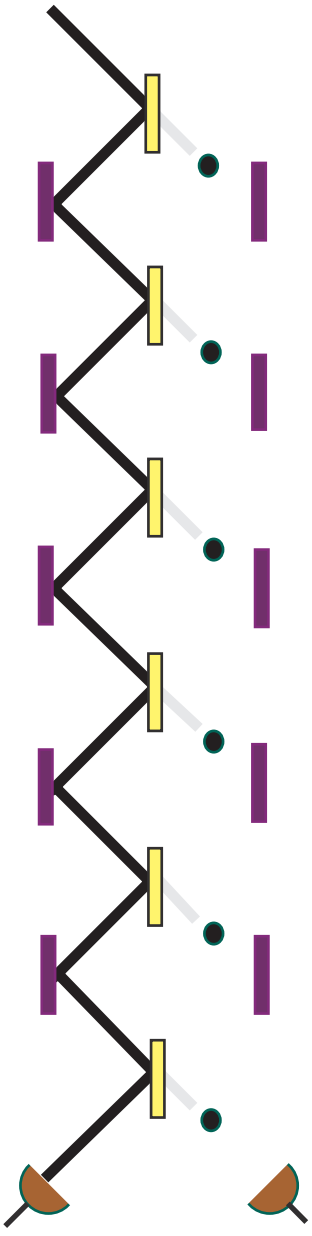

FIG.2b

Fig.2 gives a semi-classical picture of what's happening in Net1 and Net2. The darker a line representing a beam is, the more likely we are of finding the photon there. In Fig.2a, the photon probability gradually shifts from left to right. Because of the high reflectivity of the beam-splitters, this shift is very gradual. In Fig.2b, the photon probability always stays on the left side. The presence of the obstruction exerts a small influence on the photon that prevents it from straying to the right side. Note that in Fig.2a, nodes f_L and f_R both correspond physically to a detector. In Fig.2b, on the other hand, node f_L corresponds to a detector, but node f_R corresponds to the obstruction.

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EPR, Introduction

Some History

Reminiscences of a person that lived during the time when Quantum Mechanics was invented: We used to play baseball every Sunday back then. The Copenhagen interpretation was the name we gave to the personal opinions of our umpire, a guy from Copenhagen, who was fond of saying: "There are balls and strikes. But they ain't nothing till I calls them".I believe the umpire's name was Niels Bore, but don't quote me on that. Anyway, the attitude of this umpire riled Einstein to no end. It soured the game for Albert so much that he stopped playing with us on Sundays. Took to riding a wobbly bicycle instead. Sometimes Godel and I would go to his place on Sunday mornings, right before the game, to try to convince him to come with us, but he'd just say: "God", and he would point at himself as he said God, "doesn't play baseball with the Universe". This was Albert's simple way of saying that he didn't want to play baseball with us. Albert's love for the game didn't die there though...Oh no! To get back at this Bore fellow, he decided to write a paper refuting the Copenhagen interpretation. The paper described two baseballs being pitched in opposite directions. His co-authors, being baseball players too, and not wanting to get on the wrong side of the umpire, replaced the word "baseball" by "point particle" everywhere in the galley proofs, which Albert never read.

We will call an *EPR experiment* any experiment in which two or more particles originating at a common point fly far apart, and each particle is eventually intercepted and measured by an analyzer. EPR stands for Einstein, Podolsky and Rosen. These men used a thought ("gedanken") experiment of this type to discuss their objections to certain principles of the Copenhagen interpretation of Quantum Mechanics. More specifically, they objected to the principle that the reality of a physical property of a particle begins when the particle's property is measured.They recorded their grumblings in Ref.[1]. EPR considered two particles subjected to measurements of their position and momentum.

In Ref.[2], Bohm was the first to consider an EPR experiment with 2 spin 1/2 particles subjected to measurements of their spin along several directions. In Ref.[3], Bell used Bohm's experiment to propose a set of inequalities. Bell's inequalities are satisfied by a very general class of hidden variable theories called local realistic theories, but they are violated by Quantum Mechanics. In Ref.[4], Clauser-Horne proposed a set of inequalities that are very similar to those of Bell. Several laboratories

have shown fairly conclusively that the Bell-inequalities are violated by nature in precisely the way predicted by Quantum Mechanics. This proves that nature cannot be described by any local realistic theory. (It does not, however, rule out other types of hidden variable theories.)

In Ref.[5], GHZ (Greenberger, Horne and Zeilinger) proposed an EPR experiment with three spin 1/2 particles flying apart. For the GHZ experiment, Quantum Mechanics predicts that a special 3-particle correlation measurement will always yield -1 whereas Local Realism predicts that it will always yield $+1$. There is thus no need for Bell type inequalities if one wishes to use the GHZ experiment to show a difference between the predictions of Local Realism and those of Quantum Mechanics.

Local Realism, What Is It

The EPR literature often uses the term *Local Realism* without defining it precisely. We will use it to refer to a theory which (1)(Realism) models nature using classical probability and classical Bayesian nets, and (2)(Local) does not allow certain types of Bayesian nets, specifically those which have an arrow connecting two nodes which represent events with a spacelike separation. For 2 events, if (their space separation) is larger than (their time separation) times (the speed of light), then we say that there is a *spacelike separation* between the events.

EPR Paradox

Next, we will discuss briefly the objections that EPR raised in Ref.[1]. We will discuss these objections using spin 1/2 particles and spin projection measurements, even though, as we said before, EPR themselves used position and momentum measurements in their discussion.

Suppose two particles, 1 and 2, are created in a state of zero spin and thereafter they fly apart. Suppose that after they are separated by a very large distance, we measure S_{z1} (the Z component of the spin of particle 1). Any theory which satisfies angular momentum conservation will predict that if we measure S_{z1}, then a subsequent measurement of S_{z2} (the Z component of the spin of particle 2) will always yield precisely the opposite value. Since particles 1 and 2 are too far apart to influence each other, this must mean that particle 2 had a definite value for S_{z2} all along, even before S_{z1} was measured. Now suppose that we repeat the experiment, but this time we measure S_{x1} (the X component of the spin of particle 1). Again, if we measure S_{x1}, then a subsequent measurement of S_{x2} (the X component of the spin of particle 2) will always yield precisely the opposite value. Thus, again we conclude that particle 2 had a definite value for S_{x2} all along, even before S_{x1} was measured. Putting the results of these two experiments together, we are forced to conclude that a spin 1/2 particle always has simultaneously a definite value for S_x and S_z. But such states are

impossible according to Quantum Mechanics. Indeed, there is a"Heisenberg inequality" in Quantum Mechanics which says that any quantum state with a definite value for S_x will yield an unpredictable value for S_z if the latter is measured, and vice versa. EPR then argue that since Quantum Mechanics can't handle states with definite S_x and S_z, it is incomplete, it is missing some states. There must therefore be some underlying *hidden variables*, outside the realm of conventional Quantum Mechanics, that parametrize those missing states. (The term "hidden variables" was not used by EPR; it was introduced by later writers.)

Roadmap

Next we will give a roadmap to those subsequent sections of this Library that deal with EPR experiments.

- **EPR, 2 Particles, Theory**

  This section discusses the theory of EPR experiments with 2 spin 1/2 fermions. It derives the Bell inequalities for the Bohm-Bell experiment of Refs.[2]-[3] and for the Clauser-Horne experiment of Ref.[4].

- **EPR, 2 Particles, Practice**

  This section puts the results of the section "EPR, 2 Particles, Theory" into practice. It uses the Quantum Fog net entitled "EPR-2fer" to show how Quantum Mechanics violates the Bell inequalities that were derived in the section "EPR, 2 Particles, Theory".

- **EPR, 3 Particles, Theory**

  This section discusses the theory of the EPR experiment with 3 spin 1/2 fermions that was proposed by GHZ in Ref.[5].

- **EPR, 3 Particles, Practice**

  This section puts the results of the section "EPR, 3 Particles, Theory" into practice. It uses the Quantum Fog net entitled "EPR-3fer" to show how Quantum Mechanics violates local realistic expectations.

EPR experiments can also be done with particles that are not spin 1/2 fermions. For example, they can be done with photons, which are spin-1 bosons. Even though there are no text sections discussing these nets, we have also included Quantum Fog nets for EPR experiments with two photons. The photonic (optical) experiments use either (1)same-polarized photons (i.e., a scalar field), or (2)differently polarized photons (i.e., a vector field). The net entitled "EPR-2pho-Scalar", based on the experiment of Ref.[6], is of the first kind. The net entitled "EPR-2pho-Vector", based on the experiment of Ref.[7], is of the second kind.

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EPR, 2 Particles, Theory

This section deals with the theory of EPR experiments in which 2 spin 1/2 fermions fly apart. We will discuss two variations of this experiment. These variations will be referred to as the Bohm-Bell and the Clauser-Horne experiments.

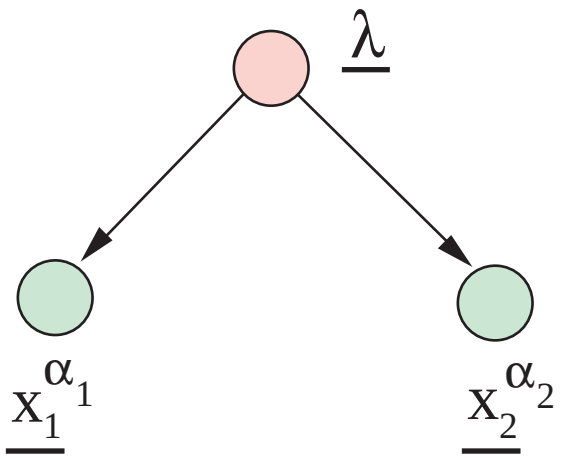


FIG.1

In local realistic theories, an EPR experiment in which 2 spin 1/2 fermions fly apart is described by the classical Bayesian net shown in Fig.1. In this figure, node $\underline{\lambda}$ represents the *hidden variables*. We will call Λ the set of states λ which node $\underline{\lambda}$ can assume. For $j \in \{1, 2\}$, node $\underline{x_j^{\alpha_j}}$ represents the outcome of a spin measurement performed on particle j. α_j represents the measurement axis. Node $\underline{x_j^{\alpha_j}}$ may assume two possible states, $+$ or $-$, depending on whether the measurement finds the spin to be pointing up or down along the α_j axis. For example, $\underline{x_1^A} = +$ if a measurement of the spin of particle 1 along the A axis yields "up".

It is convenient to define probability functions $P_j^{\alpha_j}(\cdot|\cdot)$, $P_j^{\alpha_j}(\cdot)$, $P_{12}^{\alpha_1\alpha_2}(\cdot|\cdot)$ and $P_{12}^{\alpha_1\alpha_2}(\cdot)$ as follows:

$$P_j^{\alpha_j}(x_j|\lambda) = P(\underline{x_j^{\alpha_j}} = x_j|\underline{\lambda} = \lambda) \ , \tag{1}$$

$$P_j^{\alpha_j}(x_j) = P(\underline{x_j^{\alpha_j}} = x_j) \ , \tag{2}$$

$$P_{12}^{\alpha_1\alpha_2}(x_1, x_2|\lambda) = P(\underline{x_1^{\alpha_1}} = x_1, \underline{x_2^{\alpha_2}} = x_2|\underline{\lambda} = \lambda) \ , \tag{3}$$

$$P_{12}^{\alpha_1\alpha_2}(x_1, x_2) = P(\underline{x_1^{\alpha_1}} = x_1, \underline{x_2^{\alpha_2}} = x_2) \ , \tag{4}$$

where $j \in \{1, 2\}$.

Fig.1 implies the following equation:

$$P_{12}^{\alpha_1\alpha_2}(x_1, x_2) = \sum_{\lambda\in\Lambda} P_1^{\alpha_1}(x_1|\lambda) P_2^{\alpha_2}(x_2|\lambda) P(\lambda) \ . \tag{5}$$

Because they satisfy Eq.(5), the random variables $\underline{x_1^{\alpha_1}}$ and $\underline{x_2^{\alpha_2}}$ are said to be *conditionally independent* (with respect to $\underline{\lambda}$). Note that conditionally independent variables $\underline{x_1^{\alpha_1}}$ and $\underline{x_2^{\alpha_2}}$ become independent (independent in the sense of probability theory) if the value of λ is fixed by setting $P(\lambda) = \delta(\lambda, \lambda_0)$. The acts of measuring $\underline{x_1^{\alpha_1}}$ and $\underline{x_2^{\alpha_2}}$ constitute two events. If the separation between these 2 events is spacelike, then local realistic theories require that Eq.(5) be true.

We will assume that the particles are created in a state of zero total spin angular momentum, and that they then fly apart without interacting with anything else. In Quantum Mechanics, this means that the particles are in the antisymmetric, singlet state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+_z -_z\rangle - |-_z +_z\rangle) \ . \tag{6}$$

One can show (see the appendix entitled "Spin 1/2 Particles") that for this state,

$$P_{12}^{\alpha_1\alpha_2}(++) = P_{12}^{\alpha_1\alpha_2}(--) = \frac{1}{2}\sin^2(\frac{\angle(\alpha_1, \alpha_2)}{2}) \ , \tag{7a}$$

$$P_{12}^{\alpha_1\alpha_2}(+-) = P_{12}^{\alpha_1\alpha_2}(-+) = \frac{1}{2}\cos^2(\frac{\angle(\alpha_1, \alpha_2)}{2}) \ , \tag{7b}$$

$$P_1^{\alpha_1}(+) = P_2^{\alpha_2}(+) = \frac{1}{2} \ , \tag{7c}$$

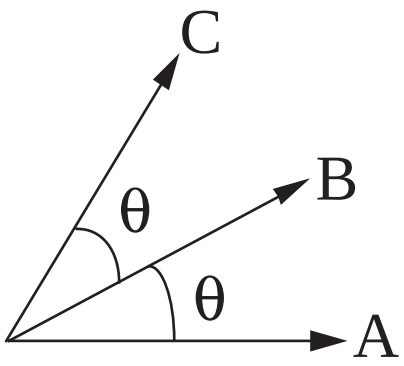
where $\angle(\alpha_1, \alpha_2)$ is the angle between axes α_1 and α_2.

Bohm-Bell Experiment

In the Bohm-Bell experiment, the spin of both particles is measured along the same 3 axes. Thus, if we call the directions of these axes A, B and C, then $\alpha_1, \alpha_2 \in \{A, B, C\}$. Suppose x, y and z are either $+$ or $-$. The Bell-inequalities for the Bohm-Bell experiment are:

$$P_{12}^{AC}(x, z) \leq P_{12}^{AB}(x, y) + P_{12}^{BC}(y, z) \ , \tag{8}$$

and the 5 other inequalities one gets by permuting the symbols A, B and C. Eq.(8) is proven in an appendix at the end of this section. The proof given in the appendix assumes that Local Realism holds.



max violation at θ = 135 °

FIG. 2

Next we will combine the local realistic result Eq.(8) with the quantum mechanical results Eqs.(7) and arrive at a contradiction. Assume axes A, B and C are coplanar and $\angle(A,B) = \angle(B,C) = \theta$ (see Fig.2). Also let $x = +, y = -$ and $z = +$ in Eq.(8). Then combining Eq.(8) with Eqs.(7) yields

$$\frac{1}{2}\sin^2(\theta) \leq \frac{1}{2}\cos^2(\frac{\theta}{2}) + \frac{1}{2}\cos^2(\frac{\theta}{2})\ . \tag{9}$$

This inequality can be simplified to

$$0 \leq 1 + \cos(2\theta) + 2\cos(\theta)\ , \tag{10}$$

which is violated (maximally) when $\theta = \frac{3\pi}{4} = 135^o$.

Thus, Quantum Mechanics tells you that if you measure the spin of particle 1 along the A axis and the spin of 2 along C, where angle(A, C) = 270 degs., and if you do this many times, you will get a probability $P_{12}^{AC}(+,+)$ that is greater than predicted by Local Realism. Somehow the particles know more about each other than one would have expected from Local Realism alone.

Clauser-Horne Experiment

In the Clauser-Horne experiment, the spin of particle 1 is measured along axes A and A' and that of particle 2 along axes B and B'. Thus, $\alpha_1 \in \{A, A'\}$ and $\alpha_2 \in \{B, B'\}$. The Bell inequalities for the Clauser-Horne experiment are:

$$0 \leq 1 + P_{12}^{AB}(++) + P_{12}^{A'B}(++) + P_{12}^{AB'}(++) - P_{12}^{A'B'}(++) - P_1^A(+) - P_2^B(+) \leq 1\ , \tag{11}$$

and the three other inequalities produced by (1)interchanging A with A', (2)interchanging B with B', (3) interchanging A with A', and B with B'. We won't present any proof of Eq.(11) here. It may be proven in various ways. See Refs.[1]-[3] if interested. The proofs given in those references assume that Local Realism holds.

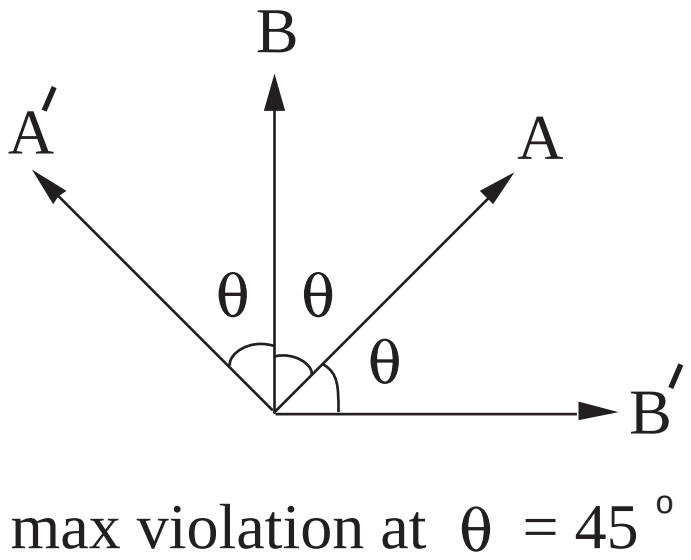


FIG. 3

Next we will combine the local realistic result Eq.(11) with the quantum mechanical results Eqs.(7) to arrive at a contradiction. Assume axes A, A', B and B' are coplanar and $\angle(B', A) = \angle(A, B) = \angle(B, A') = \theta$ (see Fig.3). Then combining Eq.(11) with Eqs.(7) yields

$$0 \leq 1 + \frac{3}{2}\sin^2(\frac{\theta}{2}) - \frac{1}{2}\sin^2(\frac{3\theta}{2}) - \frac{1}{2} - \frac{1}{2} \leq 1\ . \tag{12}$$

This last equation simplifies to

$$-2 \leq \cos(3\theta) - 3\cos(\theta) \leq 2\ , \tag{13}$$

which is violated (maximally) when $\theta = \frac{\pi}{4} = 45^o$.

Appendix: Proof Of Bell Inequalities For Bohm-Bell Experiment

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Appendix: Proof Of Bell Inequalities For Bohm-Bell Experiment

This section will present 2 proofs of the Bell Inequalities for the Bohm-Bell experiment.

For $x \in \{+,-\}$, let $\overline{x} = +$ if $x = -$, and vice versa. Hence, $\overline{x}$ is the opposite of x.

Proof 1: We begin by noticing that since the initial state must have zero spin angular momentum, one must have

$$P_1^\alpha(x|\lambda) = P_2^\alpha(\overline{x}|\lambda) \ , \tag{1}$$

where $\alpha \in \{A, B, C\}$ and $x \in \{+,-\}$. In other words, if we measure the spin of both particles along the same axis, we expect that the two measurements will always be opposite. This should be true in any theory that conserves angular momentum.

One has

$$\begin{array}{l} P_{12}^{AC}(x,z|\lambda) \\ = P_1^A(x|\lambda)P_2^C(z|\lambda) \\ = P_1^A(x|\lambda)[P_1^B(\overline{y}|\lambda) + P_1^B(y|\lambda)]P_1^C(\overline{z}|\lambda) \\ \leq \\ P_1^A(x|\lambda)P_1^B(\overline{y}|\lambda) + P_1^B(y|\lambda)P_1^C(\overline{z}|\lambda) \\ = P_{12}^{AB}(x,y|\lambda) + P_{12}^{BC}(y,z|\lambda) \end{array} \ , \tag{2}$$

where Eq.(1) has been used repeatedly. Multiplying both sides of inequality Eq.(2) by $P(\lambda)$ and adding over all $\lambda \in \Lambda$ yields

$$P_{12}^{AC}(x,z) \leq P_{12}^{AB}(x,y) + P_{12}^{BC}(y,z) \ . \tag{3}$$

Proof 2: This proof is based on the following equation from the section entitled "EPR, 2 Particles, Theory":

$$P_{12}^{\alpha_1\alpha_2}(x_1,x_2) = \sum_{\lambda\in\Lambda} P_1^{\alpha_1}(x_1|\lambda)P_2^{\alpha_2}(x_2|\lambda)P(\lambda) \ . \tag{4}$$

Suppose x_j^α is a random variable with values $x_j^\alpha \in \{+,-\}$, where $j \in \{1,2\}$ and $\alpha \in \{A, B, C\}$. x_j^α represents the value obtained by a measurement of particle j along axis α. Define X_1, X_2 and X by

$$X_1 = (x_1^A, x_1^B, x_1^C) \ , \quad X_2 = (x_2^A, x_2^B, x_2^C) \ , \quad X = (X_1, X_2) \ , \tag{5}$$

and define $\underline{X}_1, \underline{X}_2$ and $\underline{X}$ analogously.

Suppose we replace in Eq.(5) the hidden variables $\underline{\lambda}$ by the special hidden variables $\underline{X}$:

$$P(\underline{\lambda} = \lambda) \to P(\underline{X} = X) \ . \tag{6}$$

According to Quantum Mechanics, the probability distribution $P(X)$ does not exist, because its existence would imply that one can know precisely and simultaneous the values of complementary variables such as x_1^A and x_1^B. However, Local Realism, which is what we are assuming in this proof, does not object to the existence of $P(X)$.

Since the variables X arise so naturally in this problem, we will call them the *canonical hidden variables* for this problem. It might seem that we loose generality by considering only canonical hidden variables, but this is not so. When the hidden variables are not the canonical ones, their effect on this particular problem can always be mimicked identically by a suitable probability distribution $P(X)$ of the canonical hidden variables.

Notice that because of conservation of angular momentum, $P(X)$ vanishes unless $X_2 = -X_1$. Therefore, $P(X)$ can be expressed as

$$P(X) = \sigma(X_1)\delta(X_1, -X_2) \ , \tag{7}$$

where $\sigma(\cdot)$ is some probability function of X_1.

Combining Eq.(7) and Eq.(5), one gets

$$P_{12}^{AB}(x, y) \leq \sum_{x_1^C \in \{+,-\}} \sigma(x, \overline{y}, x_1^C) = \sigma(x, \overline{y}, \overline{z}) + \sigma(x, \overline{y}, z) \ , \tag{8a}$$

$$P_{12}^{BC}(y, z) \leq \sum_{x_1^A \in \{+,-\}} \sigma(x_1^A, y, \overline{z}) = \sigma(x, y, \overline{z}) + \sigma(\overline{x}, y, \overline{z}) \ , \tag{8b}$$

$$P_{12}^{AC}(x, z) \leq \sum_{x_1^B \in \{+,-\}} \sigma(x, x_1^B, \overline{z}) = \sigma(x, y, \overline{z}) + \sigma(x, \overline{y}, \overline{z}) \ . \tag{8c}$$

The first term on the right side of Eq.(8c) is the first term on the right side of Eq.(8b). The second term on the right side of Eq.(8c) is the first term on the right side of Eq.(8a). Therefore, Eq.(3) above follows.

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EPR, 2 Particles, Practice

This section deals with EPR experiments in which 2 spin 1/2 fermions fly apart. We will discuss two variations of this experiment. These variations will be referred to as the Bohm-Bell and the Clauser-Horne experiments. The Bell inequalities for both of these experiments were derived in the section entitled "EPR, 2 Particles, Theory". In this section, we use Quantum Fog to verify that Quantum Mechanics violates the Bell inequalities for both of these experiments.

Open the Quantum Fog file entitled "EPR-2fer".

- STATES: Node *Origin* has two states *ud* and *du*. *ud* is the state in which the Z spin component of particle 1 is up (+) and that of particle 2 is down (−). State *du* is defined analogously. Node $N1$ has two states: (1,0) and (0,1). State $(1, 0)$ (ditto, $(0, 1)$) corresponds to the case that particle 1 has spin − (ditto, +) when it passes through $N1$. Node $N2$ is analogous to node $N1$, but for particle 2 instead of 1. Node $X1$ has two states: (1,0) and (0,1). State $(1, 0)$ (ditto, $(0, 1)$) corresponds to the case that a measurement of particle 1's spin-projection along a particular axis, not necessarily the Z axis, yields − (ditto, +). Node $X2$ is analogous to node $X1$, but for particle 2 instead of 1.

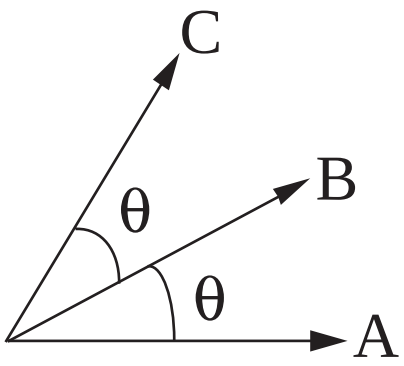
- AMPLITUDES: For node *Origin*, the amplitude of state *ud* is $\frac{1}{\sqrt{2}}$, and that of state *du* is $\frac{-1}{\sqrt{2}}$. The amplitudes of deterministic nodes $N1$ and $N2$ are self-explanatory.

  The amplitudes of node $X1$ (ditto, $X2$) can be obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info** window. We will take the quantization direction of the parent node(i.e., of node *Origin*) to be the positive Z direction. The output quantization direction of node $X1$ (ditto, $X2$) will be specified below.

- INACTIVE STATES: Throughout the experiment, all node states will be active.

- BI-NODES OF INTEREST: In the **Bi-nodes of Interest** window, we've selected the node pair $X1, X2$.

Bohm-Bell Experiment



max violation at θ = 135 °

FIG. 1

Perform the following sequence of steps with net "EPR-2fer":

1. In the **Node Prior-Info** window, press the **Generate Amplitudes...** button for node $X1$ and then for node $X2$. Use the following output quantization directions:

$$\begin{array}{lll} X1: & \theta = 0, & \phi = 0 \\ X2: & \theta = 270^o & \phi = 0 \end{array} . \tag{1}$$

   Now when you press **Go_Forward** and look at the **Bi-node Probs** window, you should find that

$$P[X1 = +, X2 = +] = \frac{1}{2}\sin^2(\frac{270^o}{2}) = 0.25 . \tag{AC}$$

2. In the **Node Prior-Info** window, press the **Generate Amplitudes...** button for node $X1$ and then for node $X2$. Use the following output quantization directions:

$$\begin{array}{lll} X1: & \theta = 0, & \phi = 0 \\ X2: & \theta = 135^o & \phi = 0 \end{array} . \tag{2}$$

   Now when you press **Go_Forward** and look at the **Bi-node Probs** window, you should find that

$$P[X1 = +, X2 = -] = \frac{1}{2}\cos^2(\frac{135^o}{2}) = 0.0732 . \tag{AB}$$

3. In the **Node Prior-Info** window, press the **Generate Amplitudes...** button for node $X1$ and then for node $X2$. Use the following output quantization directions:

$$\begin{array}{lll} X1: & \theta = 135^o, & \phi = 0 \\ X2: & \theta = 270^o & \phi = 0 \end{array} . \tag{3}$$

Now when you press **Go_Forward** and look at the **Bi-node Probs** window, you should find that

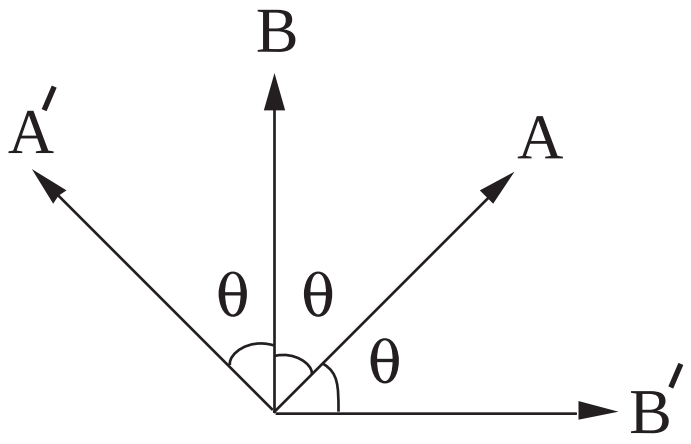
$$P[X1 = -, X2 = +] = \frac{1}{2}\cos^2(\frac{135^o}{2}) = 0.0732 . \tag{BC}$$

4. Local Realism requires the following Bell-inequality:

$$P[\text{Eq.}(AC)] \leq P[\text{Eq.}(AB)] + P[\text{Eq.}(BC)] , \tag{4}$$

where by $P[\text{Eq.}(.)]$ we mean the probability that was calculated in Eq.(.). Clearly, Eq.(4) is not satisfied by net "EPR-2fer".

Clauser-Horne Experiment



max violation at $\theta = 45^o$

FIG. 2

Perform the following sequence of steps with net "EPR-2fer":

1. In the **Node Prior-Info** window, press the **Generate Amplitudes...** button for node $X1$ and then for node $X2$. Use the following output quantization directions:

$$\begin{array}{ll} X1: & \theta = 45^o, \quad \phi = 0 \\ X2: & \theta = 90^o \quad \phi = 0 \end{array} . \tag{5}$$

Now when you press **Go_Forward** and look at the **Bi-node Probs** window, you should find that

$$P[X1 = +, X2 = +] = \frac{1}{2}\sin^2(\frac{45^o}{2}) = 0.0732 . \tag{AB}$$

2. In the **Node Prior-Info** window, press the **Generate Amplitudes...** button for node $X1$ and then for node $X2$. Use the following output quantization directions:

$$\begin{array}{ll} X1: & \theta = 135^o, \quad \phi = 0 \\ X2: & \theta = 90^o \quad \phi = 0 \end{array} . \tag{6}$$

Now when you press **Go_Forward** and look at the **Bi-node Probs** window, you should find that

$$P[X1 = +, X2 = +] = \frac{1}{2}\sin^2(\frac{45^o}{2}) = 0.0732 . \tag{A'B}$$

3. In the **Node Prior-Info** window, press the **Generate Amplitudes...** button for node $X1$ and then for node $X2$. Use the following output quantization directions:

$$\begin{array}{ll} X1: & \theta = 45^o, \quad \phi = 0 \\ X2: & \theta = 0 \quad \phi = 0 \end{array} . \tag{7}$$

Now when you press **Go_Forward** and look at the **Bi-node Probs** window, you should find that

$$P[X1 = +, X2 = +] = \frac{1}{2}\sin^2(\frac{45^o}{2}) = 0.0732 . \tag{AB'}$$

4. In the **Node Prior-Info** window, press the **Generate Amplitudes...** button for node $X1$ and then for node $X2$. Use the following output quantization directions:

$$\begin{array}{ll} X1: & \theta = 135^o, \quad \phi = 0 \\ X2: & \theta = 0 \quad \phi = 0 \end{array} . \tag{8}$$

Now when you press **Go_Forward** and look at the **Bi-node Probs** window, you should find that

$$P[X1 = +, X2 = +] = \frac{1}{2}\sin^2(\frac{135^o}{2}) = 0.4268\ . \qquad \text{(A'B')}$$

5. Local Realism requires the following Bell-inequality:

$$0 \leq P[\text{Eq.}(AB)] + P[\text{Eq.}(A'B)] + P[\text{Eq.}(AB')] - P[\text{Eq.}(A'B')] \leq 1\ , \qquad (9)$$

where by $P[\text{Eq.}(.)]$ we mean the probability that was calculated in Eq(.). Clearly, Eq.(9) is not satisfied by net "EPR-2fer".

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EPR, 3 Particles, Theory

This section deals with the theory of a particular EPR experiment with 3 spin 1/2 fermions that was first proposed by GHZ. For the GHZ experiment, Quantum Mechanics predicts that a special 3-particle correlation measurement will always yield -1 whereas Local Realism predicts that it will always yield $+1$. Proving this statement is the main goal of this section.

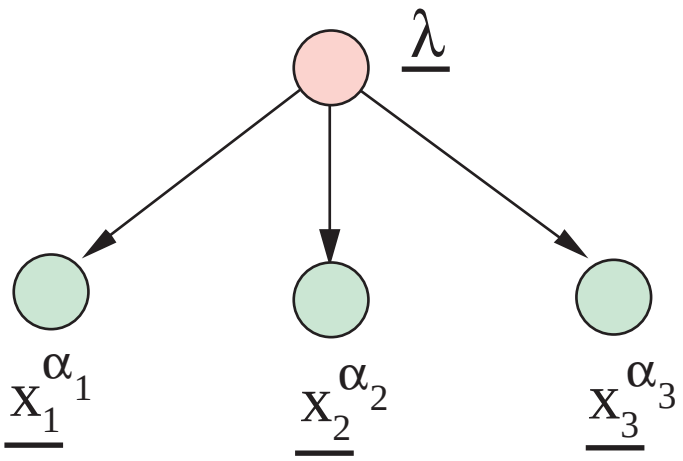

FIG.1

In local realistic theories, an EPR experiment in which 3 spin 1/2 fermions fly apart is described by the classical Bayesian net shown in Fig.1. In this figure, node $\underline{\lambda}$ represents the *hidden variables*. We will call Λ the set of states λ which node $\underline{\lambda}$ can assume. For $j \in \{1, 2, 3\}$, node $\underline{x_j^{\alpha_j}}$ represents the outcome of a spin measurement performed on particle j. α_j represents the measurement axis. Node $\underline{x_j^{\alpha_j}}$ may assume two possible states, $+$ or $-$, depending on whether the measurement finds the spin to be pointing up or down along the α_j axis. For example, $\underline{x_1^A} = +$ if a measurement of the spin of particle 1 along the A axis yields "up".

In analogy with the section entitled "EPR, 2 Particles, Theory", Fig.1 above implies

$$P_{123}^{\alpha_1\alpha_2\alpha_3}(x_1, x_2, x_3) = \sum_{\lambda\in\Lambda} P_1^{\alpha_1}(x_1|\lambda) P_2^{\alpha_2}(x_2|\lambda) P_3^{\alpha_3}(x_3|\lambda) P(\lambda) \ . \qquad (1)$$

Suppose $m_j^x, m_j^y \in \{+,-\}$ for $j \in \{1,2,3\}$. Let $M_j = (m_j^x, m_j^y)$ and $M = (M_1, M_2, M_3)$. We will restrict our measurement axes α_j to be either $\hat{x}$ (unit vector along the X axis) or $\hat{y}$ (unit vector along the Y axis). Define the function $\sigma(M)$ by

$$\sigma(M) = P(\underline{x_1}^{\hat{x}} = m_1^x, \underline{x_1}^{\hat{y}} = m_1^y, \underline{x_2}^{\hat{x}} = m_2^x, \underline{x_2}^{\hat{y}} = m_2^y, \underline{x_3}^{\hat{x}} = m_3^x, \underline{x_3}^{\hat{y}} = m_3^y) \ . \tag{2}$$

One can replace the elements of Eq.(1) as follows:

$$\lambda \to M \ , \tag{3a}$$

$$P(\lambda) \to \sigma(M) \ , \tag{3b}$$

$$P_j^{\alpha_j}(x_j|\lambda) \to \delta(x_j, m_j^{\alpha_j}) \ . \tag{3c}$$

After these replacements, Eq.(1) becomes

$$P_{123}^{\alpha_1\alpha_2\alpha_3}(x_1, x_2, x_3) = \sum_M \delta(x_1, m_1^{\alpha_1})\delta(x_2, m_2^{\alpha_2})\delta(x_3, m_3^{\alpha_3})\sigma(M) \ , \tag{4}$$

where M is summed over all its possible values (i.e., all the elements of the six fold product of set $\{+,-\}$).

According to Quantum Mechanics, the probability distribution $\sigma(M)$ does not exist, because its existence would imply that one can know precisely and simultaneous the values of complementary variables such as m_1^x and m_1^y. However, Local Realism, which is what we assumed to prove Eq.(4), does not object to the existence of $\sigma(M)$.

In going from Eq.(1) to Eq.(4), we've replaced the hidden variables λ by a special set of hidden variables M. Since the variables M arise so naturally in this problem, we will call them the *canonical hidden variables* for this problem. It might seem that we loose generality by considering only canonical hidden variables, but this is not so. When the hidden variables are not the canonical ones, their effect on this particular problem can always be mimicked identically by a suitable probability distribution $\sigma(M)$ of the canonical hidden variables.

Suppose that we assume that the 3 particles are created in the following quantum mechanical state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|+_z +_z +_z\rangle - |-_z -_z -_z\rangle) \ . \tag{5}$$

One can show (see the appendix entitled "Spin 1/2 Particles") that for this state,

$$P_{123}^{\hat{x}\hat{y}\hat{y}}(x_1, x_2, x_3) \propto \delta(x_1x_2x_3, +) \ , \tag{6a}$$

$$P_{123}^{\hat{y}\hat{x}\hat{y}}(x_1, x_2, x_3) \propto \delta(x_1x_2x_3, +) \ , \tag{6b}$$

$$P_{123}^{\hat{y}\hat{y}\hat{x}}(x_1, x_2, x_3) \propto \delta(x_1 x_2 x_3, +) \ , \tag{6c}$$

$$P_{123}^{\hat{x}\hat{x}\hat{x}}(x_1, x_2, x_3) \propto \delta(x_1 x_2 x_3, -) \ , \tag{6d}$$

where the symbol "$\propto$" means "proportional to", where $x_j \in \{+, -\}$ for $j \in \{1, 2, 3\}$. In other words, if one measures one particle along the $\hat{x}$ axis and the other two along the $\hat{y}$ axis, then the product of the 3 measurements is always $+$. However, if one measures all 3 particles along the $\hat{x}$ axis, then the product of the measurements is always $-$.

Note that to prove Eq.(4), we assumed that Local Realism holds. Next we will combine the local realistic result Eq.(4) with the quantum mechanical results Eqs.(6) to arrive at a contradiction. The local realistic equivalent of the state given by Eq.(5) would have to obey

$$\sigma(M) \propto \delta(m_1^x m_2^y m_3^y, +)\delta(m_1^y m_2^x m_3^y, +)\delta(m_1^y m_2^y m_3^x, +) \propto \delta(m_1^x m_2^x m_3^x, +) \ . \tag{7}$$

This last result and Eq.(4) would then imply that

$$P_{123}^{\hat{x}\hat{x}\hat{x}}(x_1, x_2, x_3) \propto \delta(x_1 x_2 x_3, +) \ , \tag{8}$$

which contradicts Eq.(6d). In other words, Local Realism requires that if we measure all 3 particles along the $\hat{x}$ axis, the product of these measurements should be $+$, the opposite of what Quantum Mechanics predicts.

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EPR, 3 Particles, Practice

This section deals with the particular EPR experiment with 3 spin 1/2 fermions that was first proposed by GHZ. For the GHZ experiment, Quantum Mechanics predicts that a special 3-particle correlation measurement will always yield -1 whereas Local Realism predicts that it will always yield $+1$. This statement was proven in the section entitled "EPR, 3 Particles, Theory". In this section we will use Quantum Fog to verify the quantum mechanical prediction.

Open the Quantum Fog file entitled "EPR-3fer".

- STATES: Node *Origin* has two states *uuu* and *ddd*. *uuu* is the state in which the Z spin component of all 3 particles is up (+), and state *ddd* is the state in which the Z spin component of all 3 particles is down (−). Node $N1$ has two states: (1,0) and (0,1). State $(1, 0)$ (ditto, $(0, 1)$) corresponds to the case that particle 1 has spin − (ditto, +) when it passes through $N1$. Nodes $N2$ and $N3$ are analogous to node $N1$, but for particles 2 and 3 instead of 1. Node $X1$ has two states: (1,0) and (0,1). State $(1, 0)$ (ditto, $(0, 1)$) corresponds to the case that a measurement of particle 1's spin-projection along the X axis yields − (ditto, +). Nodes $X2$ and $X3$ are analogous to node $X1$, but for particles 2 and 3 instead of 1.

- AMPLITUDES: For node *Origin*, the amplitude of state *uuu* is $\frac{1}{\sqrt{2}}$, and that of state *ddd* is $\frac{-1}{\sqrt{2}}$. The amplitudes of deterministic nodes $N1, N2$ and $N3$ are self-explanatory.

  The amplitudes of node $X1$ (ditto, $X2, X3$) were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info** window. We took the quantization direction of the parent node (i.e., of node *Origin*) to be the positive Z direction. The output quantization direction of node $X1$ (ditto, $X2, X3$) was taken to be the positive X direction. This yielded the following amplitudes, which are in agreement with the formulas in the section entitled "Spin 1/2 Particles".

$$\begin{array}{ll} \langle +_x|+_z\rangle = \cos(\frac{90^o}{2}) = \frac{1}{\sqrt{2}}, & \langle +_x|-_z\rangle = \sin(\frac{90^o}{2}) = \frac{1}{\sqrt{2}} \\ \langle -_x|+_z\rangle = \frac{-1}{\sqrt{2}}, & \langle -_x|-_z\rangle = \frac{1}{\sqrt{2}} \end{array} . \qquad (1)$$

- INACTIVE STATES: Throughout the experiment, all node states will be active.

The net has 8 possible endings. There are 2 stories with each ending, one that starts with state *uuu* and another that starts with state *ddd*.

Node *Origin* contributes a factor of either plus or minus $\frac{1}{\sqrt{2}}$ to the amplitude of each story. Nodes $N1, N2$ and $N3$ contribute a factor of 1 to each possible (i.e. non-vanishing) story amplitude. According to Eqs.(1), nodes $X1, X2$ and $X3$ contribute a factor of either plus or minus $\frac{1}{\sqrt{2}}$ to each amplitude. It quickly becomes obvious that all possible stories have an *Amp* equal to:

$$Amp = \sigma \left[\frac{1}{\sqrt{2}}\right]^4 = \sigma(0.25) \ , \tag{2a}$$

where the sign σ is defined by

$$\sigma = \left\{ \begin{array}{ll} +1 & \text{if } \ Origin = uuu \\ -1 & \text{if } \ Origin = ddd \end{array} \right\} \times (-1)^{(\text{number of } \langle -_x | +_z \rangle \text{ factors})} \ . \tag{2b}$$

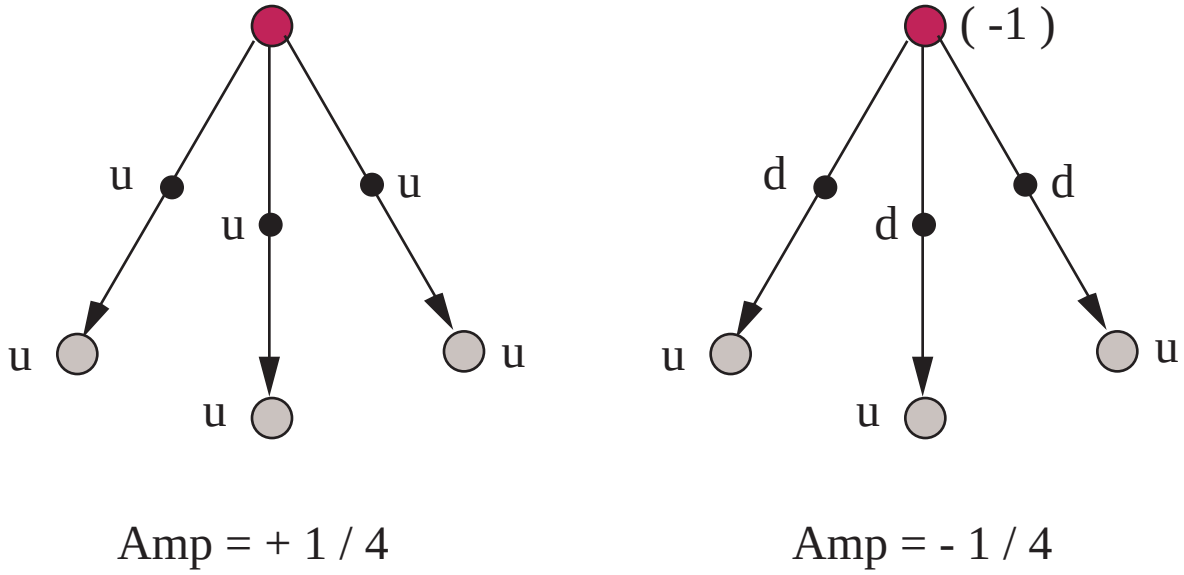

FIG.1

Fig.1 shows the two stories with ending *uuu* (i.e., $X1 = X2 = X3 = u$). The figure also shows each story's amplitude according to Eqs.(2). The two stories have opposite amplitudes and thus they interfere destructively, cancelling each other out.

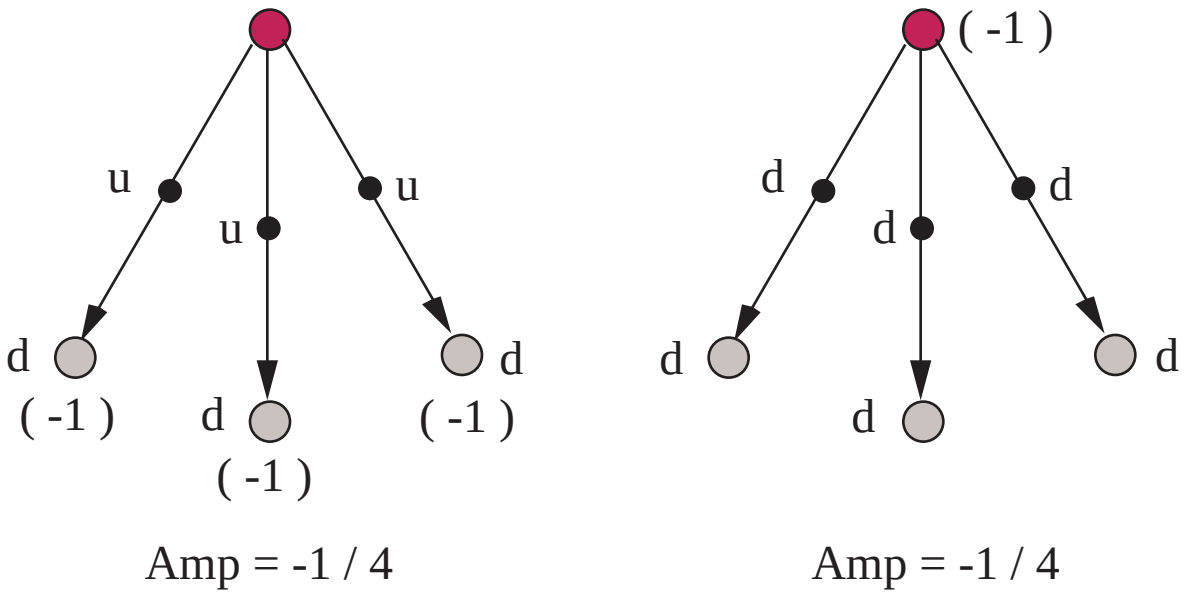

FIG.2

Fig.2 shows the two stories with ending ddd (i.e., $X1 = X2 = X3 = d$). The figure also shows each story's amplitude according to Eqs.(2). The two stories have the same amplitude and thus they interfere constructively.

Just as in Figs.1 and 2, it can be shown that for each of the following 4 endings, their two stories have opposite amplitudes:

$$uuu, udd, dud, ddu \ . \tag{3}$$

Hence, according to Quantum Mechanics, the four endings of Eq.(3) cannot occur.

Likewise, it can be shown that for each of the following 4 endings, their two stories have equal amplitudes:

$$ddd, duu, udu, uud \ . \tag{4}$$

Hence, according to Quantum Mechanics, the four endings of Eq.(4) do occur.

Define m_j for $j \in \{1, 2, 3\}$ to equal $+1$ if $Xj = u$ and -1 if $Xj = d$. Define $corr$ to be the product $m_1 m_2 m_3$. The endings of Eq.(3) all have $corr = +1$, and those of Eq.(4) all have $corr = -1$.

Thus, Quantum Mechanics predicts that only endings with $corr = -1$ can occur. And according to the section entitled "EPR, 3 Particles, Theory", Local Realism predicts exactly the opposite, that only $corr = +1$ can occur.

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Can't Clone Single Copy Of An Unknown Quantum State

As pointed out in Ref.[1], it is impossible to build a device that can clone a single copy of an unknown quantum state. This is why. Suppose that such a device exists and that the device acts on 2 particles. One particle, the "original" particle whose state we want to duplicate, is in a state $|x\rangle_{or.}$ which is unknown to us. The other particle, the "mime" particle we wish to put into the same state as the original particle, is initially in a known state $|0\rangle_{mime}$. Let operator U represent the action of the device. Then the device must do the following:

$$U|x\rangle_{or.}|0\rangle_{mime} = |x\rangle_{or.}|x\rangle_{mime} \ . \tag{1}$$

We will continue to put the state of the original particle first and that of the mime second. Suppose $|y\rangle_{or.}$ is another possible quantum state for the original particle. Suppose α and β are some arbitrary complex numbers. U must also be able to clone state $|y\rangle_{or.}$ so

$$U|y\rangle|0\rangle = |y\rangle|y\rangle \ . \tag{2}$$

Eqs.(1) and (2) and the fact that U must be a linear operator imply

$$U(\alpha|x\rangle|0\rangle + \beta|y\rangle|0\rangle) = \alpha|x\rangle|x\rangle + \beta|y\rangle|y\rangle \ . \tag{3}$$

But U must also be able to clone state $\alpha|x\rangle_{or.} + \beta|y\rangle_{or.}$ so

$$U(\alpha|x\rangle + \beta|y\rangle)|0\rangle = (\alpha|x\rangle + \beta|y\rangle)(\alpha|x\rangle + \beta|y\rangle) \ . \tag{4}$$

Eqs.(3) and (4) contradict each other, so there is no such cloning device.

If the state of a particle is known, then one can build a device to duplicate that state. If there are enough particles in the same unknown state, one can use all except one of the particles to find out what that state is. Then one can use this knowledge to build a device that duplicates the known state of the final remaining particle. But if the state of a particle is unknown and only a single copy of it is available, then there is no way of constructing a device that will always duplicate the state correctly.

References

[1] W.K. Wooters, W.H. Zurek, Nature **299**, 802 (1982) .

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Teleportation

This net represents an experiment that was first proposed in Ref.[1]. The experiment illustrates a phenomenon that the authors of Ref.[1] called "Teleportation".

In what follows, by *quantum information* we will mean information that is encoded in the quantum state of a microscopic particle. By *classical information* we will mean information that is encoded in the state of a macroscopic object.

Teleportation is best introduced by telling a tale of two cities:

It was the best of times. There existed two cities, C_1 and C_3. The scientists at C_1 had a SINGLE particle P_1 in an UNKNOWN quantum state $\Psi(P_1)$. They wished to send $\Psi(P_1)$ to C_3. They conceived of 3 plans for doing this.

- PLAN 1: Send P_1 to C_3.

  After some thought, C_1 ruled out this option because it was too risky. C_1 reasoned: If the state of P_1 were to corrupt during transit, the information it contained would be irretrievably lost.

- PLAN 2: Create a clone of P_1 and send the clone to C_3.

  C_1 wanted to prepare a second particle P_1' in the same state as P_1, without disturbing P_1 in the process. Then it could send P_1' to C_3, and keep P_1 as insurance. If the state of P_1' were to corrupt during transit, then C_1 could use P_1 to place yet another particle P_1'' in state $\Psi(P_1)$, and send P_1'' to C_3. Unfortunately, the scientists at C_1 found no way of placing P_1' in the unknown state $\Psi(P_1)$ without in the process changing the state of P_1.

- PLAN 3: Send classical information to C_3

  C_1 wanted to use its own facilities to scan P_1, extract enough classical information to characterize $\Psi(P_1)$, and transmit this classical information to C_3. C_3 could them use the classical information to place its own particle P_3 in state $\Psi(P_1)$. If the classical information were to corrupt during transit, C_1 could just retransmit it.

  The scientists at C_1 isolated P_1 from all other quantum systems, and then they scanned P_1 to extract enough information to characterize $\Psi(P_1)$. They found that performing just one measurement of P_1, no matter how gentle the measurement, put P_1 in a new state that was practically independent of $\Psi(P_1)$. They could only glean one parameter before destroying $\Psi(P_1)$. But one parameter was not enough to characterize $\Psi(P_1)$ completely.

The scientists of C_1 had all but given up when the authors of Ref.[1] pointed out a loophole that would permit plan 3 to be realized. What if P_1 were not isolated from other quantum systems before it was scanned? The authors of Ref.[1] suggested that C_3 send C_1 quantum information in the form of a particle P_2. See Fig.1 .

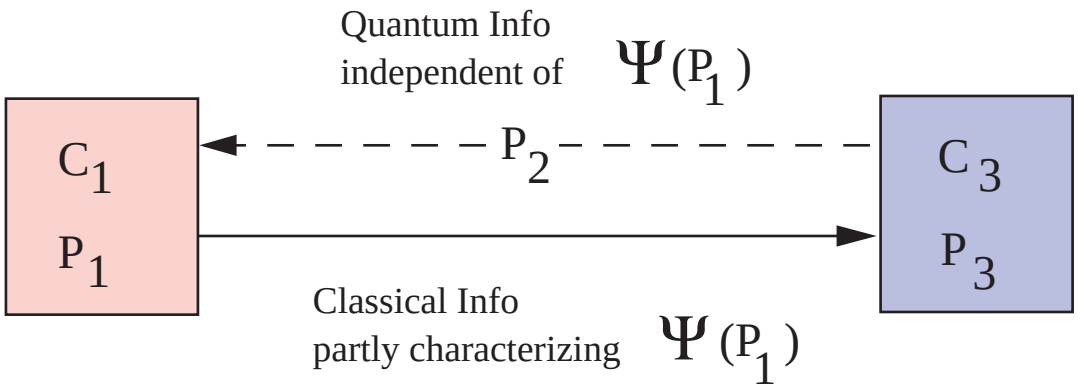

FIG. 1

Though P_2 would carry quantum information, this information would be independent of $\Psi(P_1)$. If the state of P_2 were to corrupt during transit, C_3 need only send C_1 another particle P_2' in the same state as P_2. (Since the state of P_2 would be known before it was sent to C_1, C_3 could place other particles like P_2' in the same state as P_2, even if C_3 no longer had P_2 in its possession.) Once C_1 received P_2, it could make it interact with P_1. The composite of P_1 and P_2 would be scanned to extract some classical information partly characterizing $\Psi(P_1)$. C_1 would then send this classical information to C_3. With this information, C_3 would be able to place its own particle P_3 in state $\Psi(P_1)$.

Some scientists at C_1 and C_3 understood immediately and were convinced. Others were confused and remained doubtful. Then someone used the quantum Bayesian net shown in Fig.2 to explain things.

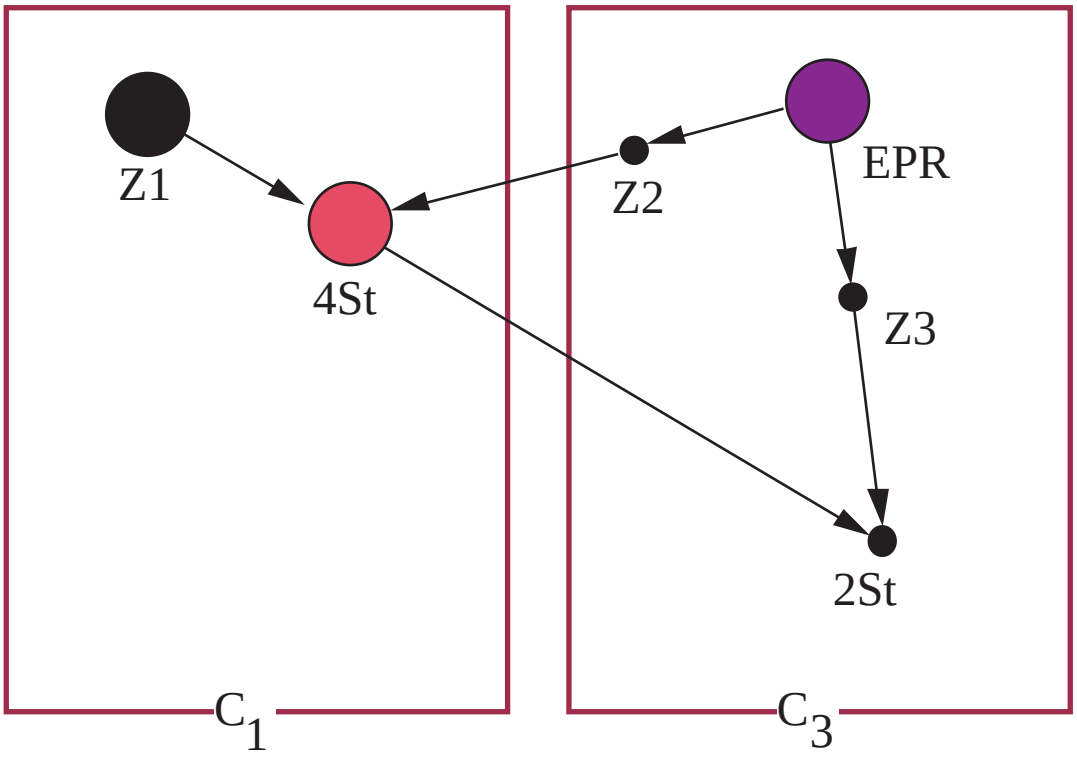

FIG. 2

And then all the scientists understood so well, and were so elated, that they called their parents, and explained Teleportation to their moms. The End.

Now open the Quantum Fog file entitled "Teleportation".

- STATES: Let u and d represent the states $|+_z\rangle$ and $|-_z\rangle$ of spin up and down along the Z direction.

  The nodes $Z1$, $Z2$ and $Z3$ are traversed by spin 1/2 particles P_1, P_2 and P_3, respectively. $Z1$, $Z2$ and $Z3$ each has 2 possible states, u and d, which represent the possible states of the particle exiting the node.

  Each state of node EPR is a product of a state for P_2 and a state for P_3. Node EPR has 4 states: u_2u_3, u_2d_3, d_2u_3 and d_2d_3. We abridge our notation by representing u_2d_3 by ud, and so on.

  Node $4St$ has four states: $\frac{1}{\sqrt{2}}(u_1u_2 \pm d_1d_2)$ and $\frac{1}{\sqrt{2}}(u_1d_2 \pm d_1u_2)$. These states represent the 4 possible outcomes of the measurement which is performed by city C_1 on the joint state of P_1 and P_2.

  Node $2St$ receives classical information from node $4St$. It receives quantum information, encoded in particle P_3, from node $Z3$. It then uses that information to prepare particle P_3 (or some other spin 1/2 particle P_3') in a special state. Node $2St$ has 2 states, u and d, which represent the possible states of the particle that exits it.

- AMPLITUDES: Let z_1, z_2, z_3, f and t represent states of nodes $Z1, Z2, Z3, 4St$ and $2St$, respectively.

  The two root nodes, $Z1$ and EPR, have the following amplitudes:

$$A_1(z_1) = \frac{1}{2}\delta(z_1, u) + \frac{\sqrt{3}}{2}\delta(z_1, d) \ , \tag{1}$$

$$A_{EPR}(z_2, z_3) = \frac{1}{\sqrt{2}}\left[\delta(z_2 z_3, ud) - \delta(z_2 z_3, du)\right] \ . \tag{2}$$

  Actually, as you'll soon see, our choice of $Z1$ amplitudes has no effect on the conclusions below.

  For $f = u_1 d_2 - d_1 u_2$, the amplitudes of nodes $4St$ and $2St$ are:

$$A_{4St}(f = u_1 d_2 - d_1 u_2 | z_1, z_2) = \frac{1}{\sqrt{2}}\left[\delta(z_1 z_2, ud) - \delta(z_1 z_2, du)\right] \ , \tag{3}$$

$$A_{2St}(t | z_3, f = u_1 d_2 - d_1 u_2) = -\delta(t, z_3) \ . \tag{4}$$

  For the 3 other possible values of f, the amplitudes of nodes $4St$ and $2St$ are given by equations similar to Eqs.(3) and (4).

- INACTIVE STATES: Initially, all states of all nodes except node $4St$ are active. For $4St$, state $u_1 d_2 - d_1 u_2$ is active and the other 3 states are inactive. Later on, you will be asked to make all node states, including those of node $4St$, active.

  Define the Feynman Integral $FI(t|f)$ by

$$FI(t|f) = \sum_{(z_2, z_3)} \sum_{z_1} A_{2St}(t | z_3, f) A_{4St}(f | z_1, z_2) A_{EPR}(z_2, z_3) A_1(z_1) \ , \tag{5}$$

where z_1 ranges over the set $\{u, d\}$ and (z_2, z_3) over $\{(u, d), (d, u)\}$. $FI(t|f)$ is the sum of the amplitudes of 4 stories. For $f = ud - du$, one can substitute Eqs.(1) to (4) into the right side of Eq.(5). Doing this, we get

$$FI(t | f = ud - du) = \frac{1}{2} A_1(t) \ . \tag{6}$$

The same technique used to prove Eq.(6) can be used to show that

$$FI(t|f) = \frac{1}{2} A_1(t) \tag{7}$$

for ANY of the four possible values of f.

The probability $P(2St = t | 4St = f)$ that $2St = t$, given that $4St = f$, is

$$P(2St = t|4St = f) = \frac{1}{k}|FI(t|f)|^2 \,, \tag{8a}$$

where

$$k = |FI(u|f)|^2 + |FI(d|f)|^2 \,. \tag{8b}$$

Plugging Eq.(7) into Eqs.(8) now yields

$$P(2St = t|4St = f) = |A_1(t)|^2 \,. \tag{9}$$

Eq.(9) tells us that regardless of the outcome of $4St$'s measurement, if $4St$ informs $2St$ through classical channels about the outcome, then $2St$ can devise an appropriate transformation given by $A_{2St}(t|f, z_3)$. If $2St$ performs this transformation, then the probabilities that the particle exiting node $2St$ be found in states $|+_z\rangle$ and $|-_z\rangle$ are the same as the probabilities that the particle exiting node $Z1$ be found in those states.

Note, however, that Eq.(9) does not guarantee that the particle exiting node $2St$ and the particle exiting node $Z1$ will have the same state vector. Suppose these state vectors are $|\Psi\rangle_1 = \alpha|+_z\rangle_1 + \beta|-_z\rangle_1$ and $|\Psi'\rangle_3 = \alpha'|+_z\rangle_3 + \beta'|-_z\rangle_3$, respectively. Assume that these state vectors are normalized. Eq.(9) implies $|\alpha| = |\alpha'|, |\beta| = |\beta'|$, but it does not imply $\alpha = \alpha', \beta = \beta'$. Eq.(7), on the other hand, does imply $\alpha = \alpha', \beta = \beta'$.

Note that it is possible to a define a Feynman Integral $FI(t)$ by

$$FI(t) = \sum_f FI(t|f) \,, \tag{10}$$

where f is summed over all its possible values. Then Eq.(7) implies

$$FI(t) = 2A_1(t) \,. \tag{11}$$

What is the probability $P(2St = t)$ that $2St$ equals t, when ALL node states, including those of node $4St$, are active (i.e., we no longer make an intermediate measurement of $4St$)? From the definition Eq.(5) of $FI(t|f)$ and the definition Eq.(10) of $FI(t)$, it is clear that we can express $P(2St = t)$ as follows:

$$P(2St = t) = \frac{1}{k}|FI(t)|^2 \,, \tag{12a}$$

where

$$k = |FI(u)|^2 + |FI(d)|^2 \,. \tag{12b}$$

Plugging Eq.(11) into Eqs.(12) now yields

$$P(2St = t) = |A_1(t)|^2 \,. \tag{13}$$

Eq.(13) tells us that even if node $4St$ is not measured, the probabilities that the particle exiting node $2St$ be found in states $|+_z\rangle$ and $|-_z\rangle$ are the same as the probabilities that the particle exiting node $Z1$ be found in those states. In fact, according to Eq.(11), even their state vectors are the same. Of course, not measuring $4St$ is equivalent to saying that $4St$ sends $2St$ quantum information instead of classical information. This quantum information might be sent via 2 spin 1/2 particles (2 qbits) or some other system in a quantum state belonging to a 4 dimensional Hilbert space.

If you select **Go Forward** in the **Change_Run-State** menu, and then you open the **Node Probs.** window, you'll see that nodes $Z1$ and $2St$ have the same probability distribution. This shows that $|\alpha| = |\alpha'|, |\beta| = |\beta'|$.

It's also possible to use Quantum Fog to confirm that $(\alpha, \beta) = e^{i\theta}(\alpha', \beta')$, where $e^{i\theta}$ is a physically insignificant phase factor. Discovering how to do this is left as an exercise to the reader. [2]

Now make all states of node $4St$ active. If you select **Go Forward** in the **Change_Run-State** menu, and then you open the **Node Probs.** window, you'll see that nodes $2St$ and $Z1$ still have the same probability distribution.

References

[1] C.H. Bennett, G.Brassard, C.Crépeau, R.Jozsa, A.Peres, W.Wootters, Phys. Rev. Lett., **70**, 1895 (1993).

[2] You'll have to extend the given net. For example, you could do the following: Add a node called $X3\_fin$ and an arrow from $2St$ to $X3\_fin$. Imagine that $2St$ sends a spin 1/2 particle P_3' to $X3\_fin$. For the states of $X3\_fin$, use the states $|+_x\rangle$ and $|-_x\rangle$ of spin up and down in the X direction. Make a second extended net, identical to the first extended net, except that instead of $X3\_fin$, you add a node called $Y3\_fin$ with the states $|+_y\rangle$ and $|-_y\rangle$ of spin up and down in the Y direction. Make yet a third extended net, identical to the first two extended nets, except that the new node is called $Z3\_fin$ and it has states $|+_z\rangle$ and $|-_z\rangle$ of spin up and down in the Z direction. The extended net with node $Z3\_fin$ allows you to prove that $|\alpha| = |\alpha'|, |\beta| = |\beta'|$. (This just verifies what the unextended net told you.) The extended net with node $X3\_fin$ allows you to prove that $\mathrm{Re}(\alpha\beta^*) = \mathrm{Re}(\alpha'\beta'^*)$. The extended net with node $Y3\_fin$ allows you to prove that $\mathrm{Im}(\alpha\beta^*) = \mathrm{Im}(\alpha'\beta'^*)$.

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Qbit Bouncing

This net represents an experiment that was first proposed in Ref.[1]. Normally, one would expect to transmit 2 cbits by transmitting 2 cbits or by transmitting 2 qbits. In this section, we will show that it is also possible to transmit 2 cbits by first receiving one qbit and then sending one qbit (i.e., by "bouncing a qbit").

Open the Quantum Fog file entitled "QbitBouncing".

- STATES: Let u and d represent the states $|+_z\rangle$ and $|-_z\rangle$ of spin up and down along the Z direction.

  The nodes $Z2$ and $Z3$ are traversed by spin 1/2 particles P_2 and P_3, respectively. $Z2$ and $Z3$ each has 2 possible states, u and d, which represent the possible states of the particle exiting the node.

  Each state of node EPR is a product of a state for P_2 and a state for P_3. Node EPR has 4 states: u_2u_3, u_2d_3, d_2u_3 and d_2d_3. We abridge our notation by representing u_2d_3 by ud, and so on.

  Node In has 4 possible states: 00, 01, 10 and 11, representing the 4 possible states of 2 cbits.

  Node $2St$ sends a spin 1/2 particle P_1 to node $4St$. Node $2St$ has 2 possible states, u and d, representing the states of P_1 after it exits the node.

  Node $4St$ has four states: $\frac{1}{\sqrt{2}}(u_1u_2 \pm d_1d_2)$ and $\frac{1}{\sqrt{2}}(u_1d_2 \pm d_1u_2)$. These states represent the 4 possible outcomes of the measurement which is performed by $4St$ on the joint state of P_1 and P_3.

- AMPLITUDES: Let z_2, z_3, t, i and f represent states of nodes $Z2, Z3, 2St, In$ and $4St$, respectively.

  The two root nodes, In and EPR, have the following amplitudes:

$$A_{In}(i) = 0.3\delta(i,00) + 0.4\delta(i,01) + 0.5\delta(i,10) + \frac{1}{\sqrt{2}}\delta(i,11)\ , \tag{1}$$

$$A_{EPR}(z_2,z_3) = \frac{1}{\sqrt{2}}\left[\delta(z_2z_3,ud) - \delta(z_2z_3,du)\right]\ . \tag{2}$$

  Actually, as you'll soon see, our choice of In amplitudes has no effect on the conclusions below.

For $i = 00$, the amplitude of node $2St$ is:

$$A_{2St}(t|i = 00, z_2) = \delta(tz_2, uu) + \delta(tz_2, dd) \ . \tag{3}$$

For the other 3 possible values of i, the amplitudes of node $2St$ are given by equations similar to Eq.(3).

The amplitude of node $4St$ is

$$A_{4St}(f|t, z_3) = \begin{cases} \frac{1}{\sqrt{2}}\left[\delta(tz_3, ud) \pm \delta(tz_3, du)\right] & \text{if } \ f = ud \pm du \\ \frac{1}{\sqrt{2}}\left[\delta(tz_3, uu) \pm \delta(tz_3, dd)\right] & \text{if } \ f = uu \pm dd \end{cases} \ . \tag{4}$$

- INACTIVE STATES: Initially, all states of all nodes except node In are active. For In, state 00 is active and the other 3 states are inactive. Later on, you will be asked to make all node states, including those of node In, active.

Define the Feynman Integral $FI(f|i)$ by

$$FI(f|i) = \sum_{(z_2,z_3)} \sum_{t} A_{4St}(f|t, z_3) A_{2St}(t|i, z_2) A_{EPR}(z_2, z_3) A_{In}(i) \ , \tag{5}$$

where t ranges over the set $\{u, d\}$ and (z_2, z_3) over $\{(u, d), (d, u)\}$. $FI(f|i)$ is the sum of the amplitudes of 4 stories. For $i = 00$, one can substitute Eqs.(1) to (4) into the right side of Eq.(5). Doing this, we get

$$FI(f|i = 00) = \delta(f, ud - du) A_{In}(i = 00) \ . \tag{6}$$

Since nodes In and $4St$ both have 4 states, it is possible to map the states of In into the states of $4St$ in a 1-1 fashion. Define the map $\phi : i \to f$ by $\phi(00) = ud - du$, $\phi(01) = ud + du$, $\phi(10) = uu - dd$ and $\phi(11) = uu + dd$. The same technique used to prove Eq.(6) can be used to show that

$$FI(f|i) = \delta(f, \phi(i)) A_{In}(i) \tag{7}$$

for ANY of the four possible values of i.

The probability $P(4St = f|In = i)$ that $4St = f$, given that $In = i$, is

$$P(4St = f|In = i) = \frac{1}{k} |FI(f|i)|^2 \ , \tag{8a}$$

with

$$k = \sum_{f} |FI(f|i)|^2 \ , \tag{8b}$$

where f is summed over its 4 possible values. Plugging Eq.(7) into Eqs.(8) now yields

$$P(4St = f|In = i) = \delta(f, \phi(i)) \; . \tag{9}$$

Eq.(9) tells us that In can send four possible messages through classical channels to $2St$, and $4St$ will be able to tell unequivocally what message In sent.

Note that it is possible to a define a Feynman Integral $FI(f)$ by

$$FI(f) = \sum_i FI(f|i) \; , \tag{10}$$

where i is summed over all its possible values. Then Eq.(7) implies

$$FI(f) = A_{In}(\phi^{-1}(f)) \; , \tag{11}$$

where ϕ^{-1} is the inverse of the function ϕ.

What is the probability $P(4St = f)$ that $4St$ equals f, when ALL node states of the net, including those of node In, are active? From the definition Eq.(5) of $FI(f|i)$ and the definition Eq.(10) of $FI(f)$, it is clear that we can express $P(4St = f)$ as follows:

$$P(4St = f) = \frac{1}{k}|FI(f)|^2 \; , \tag{12a}$$

where

$$k = \sum_f |FI(f)|^2 \; , \tag{12b}$$

where f is summed over its 4 possible values. Plugging Eq.(11) into Eqs.(12) now yields

$$P(4St = f) = |A_{In}(\phi^{-1}(f)|^2 \; . \tag{13}$$

Eq.(13) tells us that when node In sends quantum instead of classical information to $2St$, the probability distributions measured at nodes In and $4St$ are the same. In fact, according to Eq.(11), even the state vectors of the particles exiting these nodes are the same. Of course, since node In has four states, for it to send quantum information, it must do so via 2 spin 1/2 particles (2 qbits) or some other system in a quantum state belonging to a 4 dimensional Hilbert space.

Make all states of node In inactive except for state 00. Select **Go Forward** in the **Change_Run-State** menu, and then open the **Node Probs.** window. You'll see that node $4St$ is in state $ud - du$ with unit probability. Now make all states of node In inactive except for state 01. When you run the net, you'll see that this time node $4St$ is in state $ud + du$ with unit probability. And so on. There is a 1-1 map between the single state of In which you keep active, and the single state of $4St$ which occurs with unit probability.

Now make all states of node In active. Select **Go Forward** in the **Change_Run-State** menu, and then open the **Node Probs.** window. You'll see that nodes In and $4St$ have the same probability distribution.

References

[1] C.H. Bennett, S.J. Wiesner, Phys. Rev. Lett., **69**, 2881 (1992).

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Comparison Between Teleportation And Qbit Bouncing

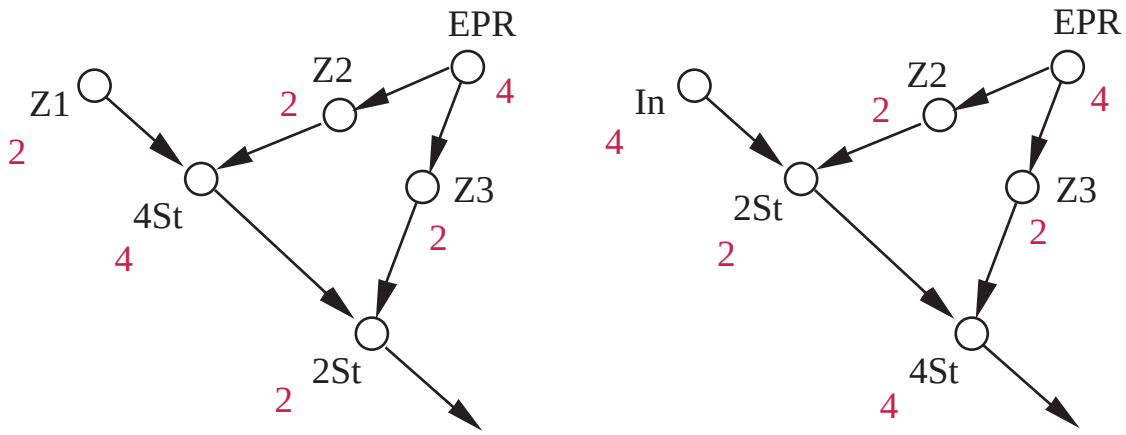
- Both nets have the same EPR triangle topology. They can both be described as EPR mediated transmission of information. However, corresponding nodes have different numbers of states. Teleportation has 2-4-2 states along its $Z1 - 4St - 2St$ line, whereas Qbit Bouncing has 4-2-4 states along its corresponding $In - 2St - 4St$ line.

- We'll say that a *node is in a fixed state* if all but one of its states is inactive. The Teleportation net has two different modes of operation that are of interest:

    1. Node $4St$ sends 2 cbits of classical information to node $2St$. Node $4St$ is in a fixed state, since that is how we model transmission of classical information.
    2. No node is in a fixed state; they all transmit quantum information.

- The Qbit Bouncing net has two different modes of operation that are of interest:

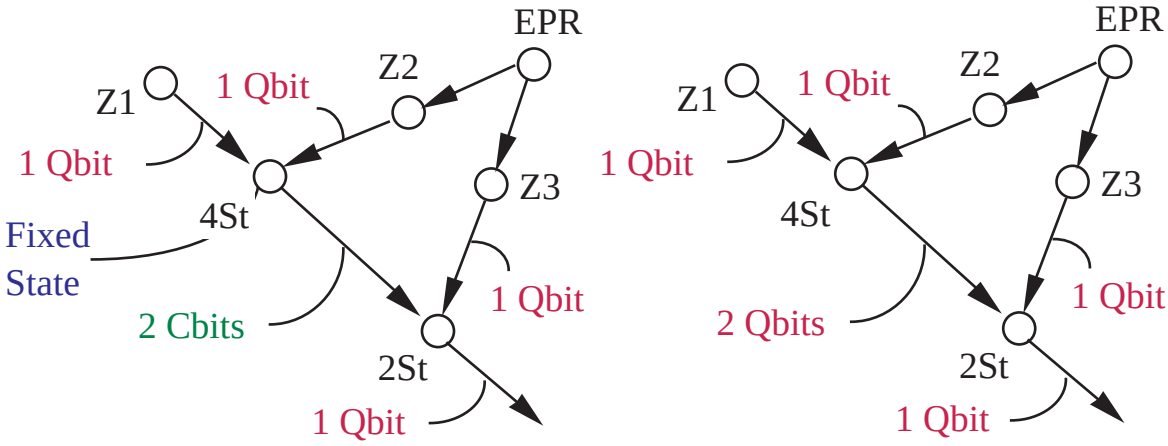
    1. Node In sends 2 cbits of classical information to node $2St$. Node In is in a fixed state.
    2. No node is in a fixed state; they all transmit quantum information.



Teleportation

Qbit Bouncing

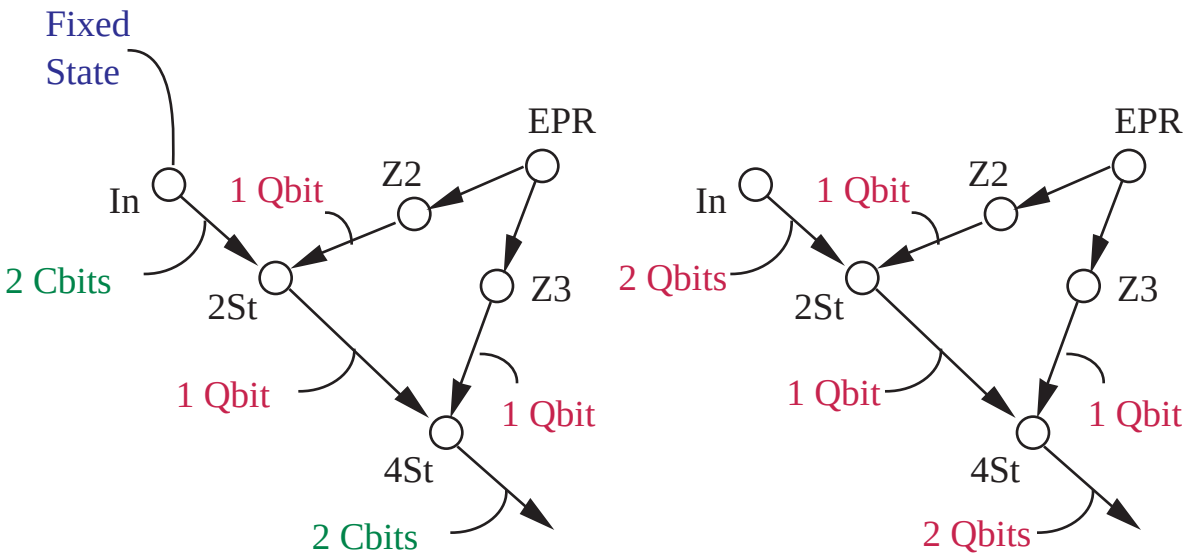
FIG.1



Teleportation Mode 1

Teleportation Mode 2

FIG.2



Qbit Bouncing Mode 1

Qbit Bouncing Mode 2

FIG.3

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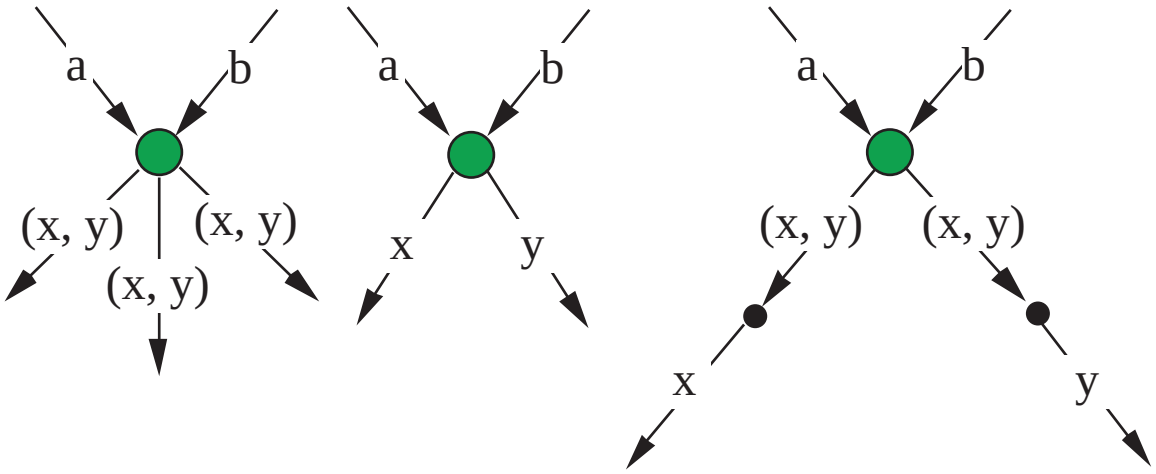
Bayesian Nets Versus Circuits

In the next few sections, we will be discussing qbit computers[1], [2]. Much of the literature about qbit computers uses quantum circuits to describe them. We will be using quantum Bayesian nets instead (although often we will present both a Bayesian net and an equivalent circuit.) Therefore, let's take a moment to compare circuits and Bayesian nets.

Just what do we mean by a circuit anyway? Like a Bayesian net, a circuit is a collection of nodes with arrows connecting certain pairs of these nodes. However, in a Bayesian net every arrow coming out of the same node carries the same information, whereas in a circuit these arrow may carry different information.



Bayesian net FIG. 1a

circuit FIG. 1b

Bayesian net FIG. 1c

For example, suppose that Fig.1a (ditto, Fig.1b) represents a Controlled-Not in a Bayesian net (ditto, in a circuit). Suppose $a, b \in \{0, 1\}$ are the states of the two input bits. Likewise, suppose $x, y \in \{0, 1\}$ are the states of the two output bits. In Fig.1a, there are 3 outgoing arrows and they all carry the same information, namely (x, y). The information (x, y) fully characterizes the state of the node. In Fig.1b, there are 2 outgoing arrows. One outgoing arrow carries x, the other y. Neither arrow carries full information about the state of the node, although together they do. In a Bayesian net, a Controlled-Not node can have any number of outgoing arrows. In a quantum circuit, it must have precisely 2 outgoing arrows.

Fig.1c shows a Bayesian net with 3 nodes (one Controlled-Not and two Marginalizers) which acts just like the circuit of Fig.1b. In general, any quantum process that can be described by a quantum circuit can also be described by a quantum Bayesian net. Given a circuit, its corresponding Bayesian net can be built by adding Marginalizer nodes to the circuit topology.

Note that everything we have said so far applies regardless of whether we are comparing quantum Bayesian nets with quantum circuits, or classical Bayesian nets with classical circuits.

Above we have compared quantum circuits with quantum Bayesian nets. How about the Feynman diagrams used in Particle Physics, how do they compare with quantum Bayesian nets? Feynman diagrams are quantum circuits, not Bayesian nets. Each Feynman diagram represents a small contribution in a perturbative series. In contrast, there is nothing intrinsically perturbative about a quantum Bayesian net.

References

[1] D. P. DiVincenzo, *Science* **270**, 255 (1995).

[2] Andrew Steane, Review To Appear in *Reports on Progress in Physics*. Also available as Los Alamos eprint quant-ph/9708022

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Nets for Generating 2- and 3-Qbit EPR States

Qbit computing deals with operations performed on an array of qbits. These operations can be represented by quantum Bayesian nets whose nodes have states that are labelled by binary numbers (like 10100 for 5 qbits). In this section, we will give two simple but useful examples of qbit computations. The first computation produces a 2-qbit EPR state. The second produces a type of 3-qbit EPR state (the so called GHZ state). We will give both a quantum Bayesian net and a quantum circuit for each computation.

Henceforth, when using a multi-qbit state vector, we will label its component single-qbit state vectors by 0, 1, 2 ..., from right to left with the rightmost state vector being labelled 0. Thus, for N_b qbits,

$$|00\cdots 0\rangle = |0\rangle_{N_b-1}\cdots|0\rangle_2|0\rangle_1|0\rangle_0 \ . \tag{1}$$

Below, we will use the symbol U_0 to represent the following operator:

$$U_0 = \frac{1}{\sqrt{2}}\left[\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right]_0 \ . \tag{2}$$

U_0 rotates qbit 0 by 90 degrees about the Y axis. See the appendix entitled "Qbits" for more information about U_0.

First, let's consider the 2-qbit EPR state.

One may build this state by starting with 2 qbits in the state $|00\rangle$, and applying U_0 followed by a controlled-not:

$$\begin{array}{l} \sigma_{x1}^{1-n_0} U_0|00\rangle \\ = \sigma_{x1}^{1-n_0}\frac{1}{\sqrt{2}}(|00\rangle - |01\rangle) \\ = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \end{array} \ . \tag{3}$$

(See the appendix entitled "Controlled-Not" for information about controlled-nots.)

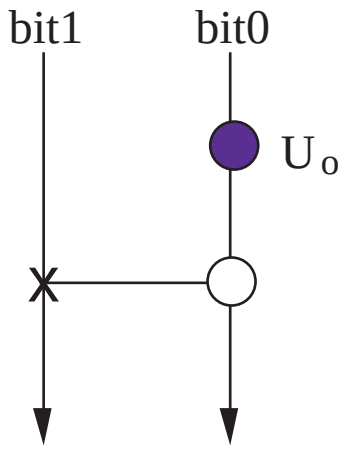

FIG. 1

Fig.1 shows the quantum circuit for this process. Note that time flows downward in this diagram. Note also that each vertical line represents one qbit. The lines are labelled 0, 1, 2 . . . from right to left with the rightmost line being labelled 0. This is the same labelling pattern that we used in Eq.(1).

Why not combine U_0 and the controlled-not into a single transformation? The literature about qbit computers often restricts the type of nodes that it uses in a quantum circuit to be either deterministic nodes (such as controlled-nots) or qbit rotations. This restriction does not represent a loss of generality. Indeed, it can be proven that for any positive integer N_b, any unitary transformation acting on states of N_b qbits can be expressed as a product of controlled-nots and qbit rotations (called "elementary gates"). This is analogous to the fact that for any classical digital circuit, one can find an equivalent circuit composed solely of NAND gates (called "elementary gates").

Now open the Quantum Fog file entitled "2QbitEPR".

- STATES: All nodes have two states, 0 and 1, except for node *cnot* which has four states, 00, 01, 10, 11.

- AMPLITUDES: Since we wish the system of 2 qbits to start in state $|00\rangle$, we've set $in1 = in0 = 0$ with unit amplitude. The amplitudes of all other nodes were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info** window. In the case of node *qrot*, we set $\theta_2 = 45^0$ and all other thetas equal to zero. In the case of node *cnot*, we used the Controlled-Not dialog for deterministic nodes. In the case of nodes $out1, out0$, we used the Marginalizer dialog for deterministic nodes.

- INACTIVE STATES: Throughout the experiment, all node states will be active.

The net has 2 possible endings, $(out1, out0) = (1, 0)$ and $(out1, out0) = (0, 1)$. There is one story with each of these endings. The 2 stories have opposite amplitudes.

Next, let's consider the 3-qbit GHZ state.

One may build this state by starting with 3 qbits in the state $|000\rangle$, and applying U_0 followed by two controlled-nots:

$$\begin{array}{l} \sigma_{x2}^{n_1}\sigma_{x1}^{n_0}U_0|000\rangle \\ = \sigma_{x2}^{n_1}\sigma_{x1}^{n_0}\frac{1}{\sqrt{2}}(|000\rangle - |001\rangle) \\ = \sigma_{x2}^{n_1}\frac{1}{\sqrt{2}}(|000\rangle - |011\rangle) \\ = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) \end{array} \quad . \tag{4}$$

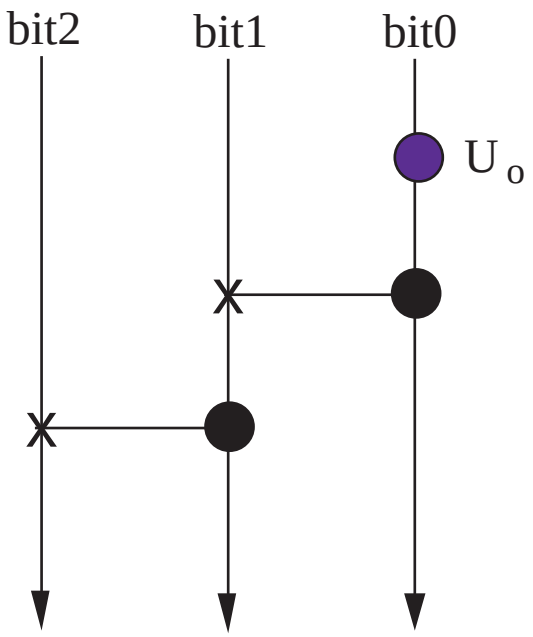

FIG. 2

Fig.2 shows the quantum circuit for this process.

Now open the Quantum Fog file entitled "3QbitEPR".

- STATES: All nodes have two states, 0 and 1, except for nodes $cnot\_a, cnot\_b$ which have four states, 00, 01, 10, 11.
- AMPLITUDES: Since we wish the system of 3 qbits to start in state $|000\rangle$, we've set $in2 = in1 = in0 = 0$ with unit amplitude. The amplitudes of all other nodes were obtained by pressing the **Generate Amplitudes...** button of the **Node Prior-Info** window. In the case of node $qrot$, we set $\theta_2 = 45^0$ and all other

thetas equal to zero. In the case of nodes $cnot\_a, cnot\_b$, we used the Controlled-Not dialog for deterministic nodes. In the case of nodes $inter, out2, out1, out0$, we used the Marginalizer dialog for deterministic nodes.

- INACTIVE STATES: Throughout the experiment, all node states will be active.

The net has 2 possible endings, $out2 = out1 = out0 = 1$ and $out2 = out1 = out0 = 0$. There is one story with each of these endings. The 2 stories have opposite amplitudes.

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Classical Error Correction - Generalities

In Classical Probability, one speaks of *coding* k cbits into n cbits, where $1 \leq k < n$. *Linear coding* is a special type of coding that works as follows. We start with an *uncoded message* v_{in} which is a k dimensional vector with components in $\{0, 1\}$. We apply to v_{in} an operator C called the *coding operator*. C is an $n \times k$ dimensional matrix with entries in $\{0, 1\}$. The result is an n dimensional vector v_{out}, given by

$$v_{out} = C v_{in} \ . \tag{1}$$

(Sums in the last equation are to be carried out in base 2). v_{out} is called the *encoded message*. The columns of C are called the *basis codewords*. A *valid codeword* is any linear combination (with coefficients in $\{0, 1\}$ and sums in base 2) of the basis codewords. The set of valid codewords is referred to as the *coding space*.

Transmission of v_{out} through a *noisy channel* introduces errors into it. The errors in v_{out} consist of random bit flips in one or more of the n bits. Multiple bit flips may be correlated. A noisy channel is characterized by the probability distribution of these random bit flips. An *error correction method* for a particular noisy channel consists of a coding operator C and a *recovery method* that allows recovery of the signal v_{in} after it is coded with C and transmitted through that noisy channel.

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Quantum Error Correction - Generalities

In *quantum coding*, we want to preserve the state vector of a *system* of k qbits, where $k \geq 1$. For simplicity, we will assume henceforth that $k = 1$. Suppose the system is qbit 0 in an array of qbits. Suppose also that initially, the system is uncorrelated to the other qbits in the array and lies in a state $|\phi_{in}\rangle$ given by

$$|\phi_{in}\rangle_{sys.} = \alpha|0\rangle_0 + \beta|1\rangle_0 = \sum_{i=0}^{1} \alpha_i |i\rangle_0 \ . \tag{1}$$

Besides the system, there are $n - k = n - 1$ qbits in the array. We will call these n-1 qbits the *coding ancilla*. Ancilla is the Latin word for servant. We call them coding ancilla to distinguish them from other groups of ancilla that will arise later on. Call the positions of the coding ancilla 1, 2 , ... Suppose each coding ancilla lies initially in state $|0\rangle$. We will call the combination of the system and coding ancilla the *extended system*. Suppose we perform on the extended system an operation described by a unitary operator C called the *coding operator*. Define $|w\rangle_{xtd.sys.}$ by

$$|w\rangle_{xtd.sys.} = C|0\cdots 0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.} \ . \tag{2}$$

Note that

$$|w\rangle_{xtd.sys.} = \sum_{i=0}^{1} \alpha_i |w_i\rangle_{xtd.sys.} \ , \tag{3}$$

where

$$|w_i\rangle_{xtd.sys.} = C|0\cdots 0\rangle_{c.anc.}|i\rangle_{sys.} \ . \tag{4}$$

We will call $|w_0\rangle$ and $|w_1\rangle$ the *basis codewords*. Note that they satisfy

$$\langle w_i|w_j\rangle = \delta_{i,j} \tag{5}$$

for $i, j \in \{0, 1\}$. $|w\rangle$ is said to be a *valid codeword*, and the vector space $\mathcal{W}$ of all $|w\rangle$ is called the *coding space*:

$$\mathcal{W} = \left\{ \sum_{i=0}^{1} \alpha_i |w_i\rangle \middle| \alpha_i \in C.N. \text{ for } i \in \{0, 1\} \right\} \ , \tag{6}$$

where $C.N.$ is the set of complex numbers.

Interaction with the environment will introduce errors into $|w\rangle_{xtd.sys.}$. Any number of qbits may be affected. Let's assume first the simplest situation, which should be valid for short enough times, that only a single qbit is affected. The most general interaction of qbit β with the environment is of the form

$$|\epsilon\rangle_{env.}|i\rangle_\beta \rightarrow |\epsilon_{i0}\rangle_{env.}|0\rangle_\beta + |\epsilon_{i1}\rangle_{env.}|1\rangle_\beta \tag{7}$$

for $i \in \{0,1\}$, where $|\epsilon_{i0}\rangle$ and $|\epsilon_{i1}\rangle$ are arbitrary states of the environment. If $\nu_i \in C.N.$ for $i \in \{0,1\}$, then Eq.(7) implies that

$$|\epsilon\rangle \left[\sum_{i=0}^{1} \nu_i |i\rangle_\beta\right] \rightarrow \sum_{i=0}^{1} \nu_i \sum_{j=0}^{1} |\epsilon_{ij}\rangle|j\rangle_\beta = \sum_{i'=0}^{1} |\epsilon_{i'j}\rangle|j\rangle_\beta\langle i'|_\beta \left[\sum_{i=0}^{1} \nu_i |i\rangle_\beta\right] . \tag{8}$$

If we represent $|0\rangle_\beta$ and $|1\rangle_\beta$ as follows

$$|0\rangle_\beta = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ |1\rangle_\beta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \tag{9}$$

then

$$|0\rangle_\beta\langle 0|_\beta = \frac{1+\sigma_{z\beta}}{2} , \tag{10a}$$

$$|0\rangle_\beta\langle 1|_\beta = \frac{\sigma_{x\beta}+i\sigma_{y\beta}}{2} , \tag{10b}$$

$$|1\rangle_\beta\langle 0|_\beta = \frac{\sigma_{x\beta}-i\sigma_{y\beta}}{2} , \tag{10c}$$

$$|1\rangle_\beta\langle 1|_\beta = \frac{1-\sigma_{z\beta}}{2} . \tag{10d}$$

Using Eqs.(10), Eq.(8) becomes

$$|\epsilon\rangle \left[\sum_{i=0}^{1} \nu_i |i\rangle_\beta\right] \rightarrow \sum_a |\epsilon_a\rangle E_a^{(\beta)} \left[\sum_{i=0}^{1} \nu_i |i\rangle_\beta\right] , \tag{11}$$

where the index a ranges over $\{1,x,y,z\}$, and $E_0^{(\beta)} = 1$, $E_x^{(\beta)} = \sigma_{x\beta}$, $E_y^{(\beta)} = \sigma_{y\beta}$, $E_z^{(\beta)} = \sigma_{z\beta}$. All that Eq.(11) is saying is that the interaction of the environment with bit β is described by a general 2×2 matrix, and such matrices can always be expressed as a linear combination of the Pauli matrices and the 2×2 identity matrix.

From the appendix entitled "Qbits", we know that $\sigma_{x\beta}$ flips β, $\sigma_{z\beta}$ may change its phase, and $\sigma_{y\beta}$ can do both. In contrast, classical bits can only be flipped by the environment.

If the environment affects more than one qbit, then Eq.(11) is no longer valid. We have instead

$$|\epsilon\rangle_{env.}|w\rangle_{xtd.sys.} \rightarrow \sum_{a=0}^{N_e} |\epsilon_a\rangle_{env.} E_a |w\rangle_{xtd.sys.} \ , \tag{12}$$

where $E_0, E_1, \ldots E_{N_e}$ are all the possible errors, including the identity operator (the no error case). Define $\mathcal{E}$ by

$$\mathcal{E} = \{E_0, E_1, \ldots E_{N_e}\} \ . \tag{13}$$

A *quantum error correction method* for a set $\mathcal{E}$ of possible errors consist of an operator C and a *recovery method*, such that

$$\sum_a |\epsilon_a\rangle_{env.} E_a C |0\rangle_{c.anc.} |\phi_{in}\rangle_{sys.} \rightarrow |\xi\rangle_{other} |\phi_{in}\rangle_{sys.} \ , \tag{14}$$

where the arrow indicates the effect of applying the recovery method. Hence, applying the recovery method converts (env. + c.anc. + sys.) into a tensor product of $|\phi_{in}\rangle$ for the system times a state for everything else. In other words, the system is left finally in a state which is uncorrelated to everything else, and which is identical to the original state of the system. We will say more about recovery methods in the next section.

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Recovery Methods For Quantum Error Correction

At the end of the previous section entitled "Quantum Error Correction - Generalities", we defined what is meant by a quantum error correction method. The last step of such correction methods is the application of a recovery method. Next we will describe 3 possible recovery methods.

Method 1: No Diagnostic Operator

This recovery method has been advocated in, for example, Ref.[1]. It consists of a single step: apply a unitary operator R. R, which we will call the *recovery operator*, has the following effect:

$$R\left\{\sum_a |\epsilon\rangle_{env.} E_a C|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\right\} = \left\{\sum_a |\epsilon\rangle_{env.}|\lambda_a\rangle_{c.anc.}\right\}|\phi_{in}\rangle_{sys.}\ . \tag{1}$$

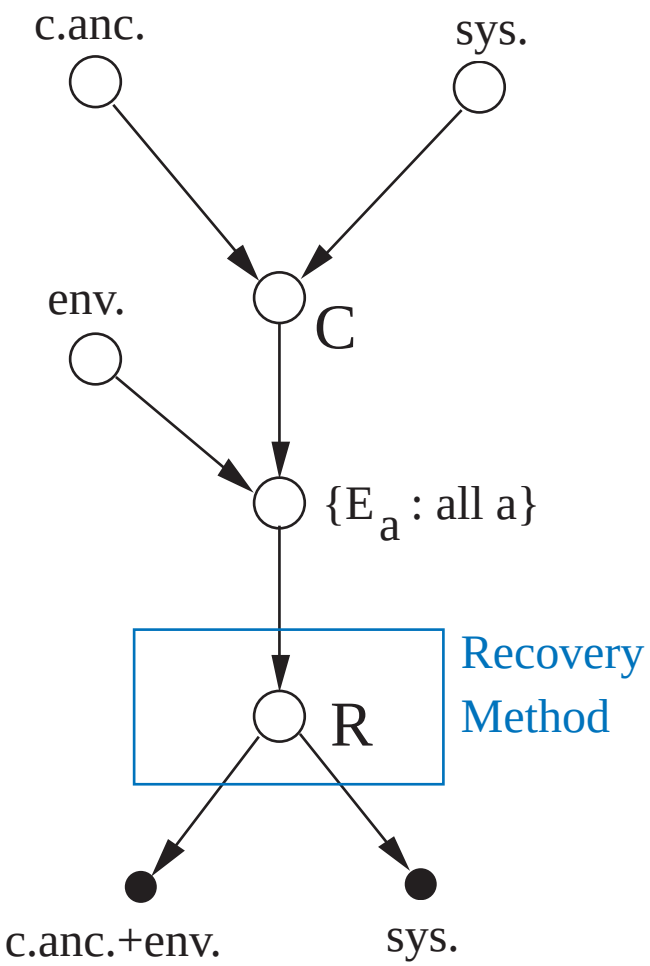

FIG. 1

Fig.1 gives a net representation of error correction using this recovery method.

Method 2: Diagnostic Operator But No Diagnostic Ancilla

This recovery method has been advocated in, for example, Ref.[2].

The first step is to apply a unitary operator D, which we will call the *diagnostic operator*, whose effect is

$$D\left\{\sum_a |\epsilon_a\rangle_{env.} E_a C|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\right\} = \sum_a |\epsilon_a\rangle_{env.}|\mu_a\rangle_{c.anc.} F_a^\dagger|\phi_{in}\rangle_{sys.} \ , \qquad (2)$$

where $F_a^\dagger$ is some unitary operator that depends on the error index a. The states $|\mu_a\rangle$ are orthonormal, but they don't necessarily span the full Hilbert space $\mathcal{H}_{c.anc.}$ of the coding ancilla.

The next step is to form a basis which includes the states $|\mu_a\rangle$ for all a, and to measure the coding ancilla in this basis. When we do, and we obtain the value μ_b, then the state of the (environment + system) becomes

$$\langle\mu_b| \left\{ \sum_a |\epsilon_a\rangle_{env.} |\mu_a\rangle_{c.anc.} F_a^\dagger |\phi_{in}\rangle_{sys.} \right\} = |\epsilon_b\rangle_{env.} F_b^\dagger |\phi_{in}\rangle_{sys.} \ . \tag{3}$$

From μ_b, we can determine the error index b. The final step is to apply the operator F_b, where b is the value that we found for the error index.

$$F_b \left\{ |\epsilon_b\rangle_{env.} F_b^\dagger |\phi_{in}\rangle_{sys.} \right\} = |\epsilon_b\rangle_{env.} |\phi_{in}\rangle_{sys.} \ . \tag{4}$$

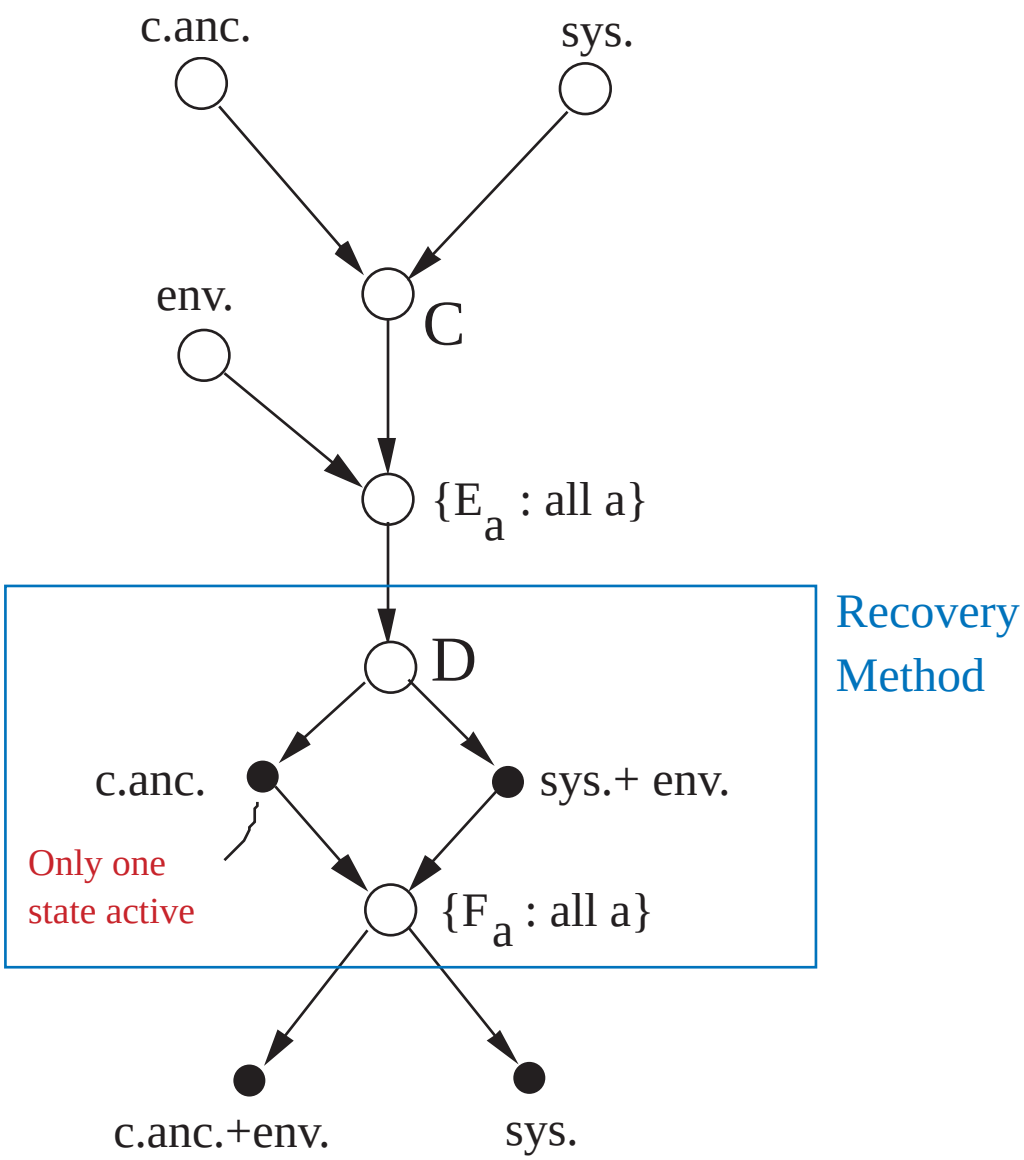

FIG. 2

Fig.2 gives a net representation of error correction using this recovery method. In this figure, the measurement of the coding ancilla corresponds to making active only one state of the measured node.

Method 3: Diagnostic Operator And Diagnostic Ancilla

In this method, we add to (environment + extended system) some *diagnostic ancilla* in state $|0\cdots 0\rangle_{d.anc.}$.

Next we apply a unitary operator D, which we will call the *diagnostic operator*, whose effect is

$$D|0\rangle_{d.anc.}\left\{\sum_a |\epsilon_a\rangle_{env.}E_aC|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\right\} = \sum_a |\epsilon_a\rangle_{env.}|\lambda_a\rangle_{d.anc.}F_a^\dagger|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\ , \tag{5}$$

where $F_a^\dagger$ is some unitary operator that depends on the error index a. The states $|\lambda_a\rangle$ are orthonormal, but they don't necessarily span the full Hilbert space $\mathcal{H}_{d.anc.}$ of the diagnostic ancilla.

The next step is to form a basis which includes the states $|\lambda_a\rangle$ for all a, and to measure the diagnostic ancilla in this basis. When we do, and we obtain the value λ_b, then the state of (env. + c.anc. + sys.) becomes

$$\langle\lambda_b|\left\{\sum_a |\epsilon_a\rangle_{env.}|\lambda_a\rangle_{d.anc.}F_a^\dagger|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\right\} = |\epsilon_b\rangle_{env.}F_b^\dagger|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\ . \tag{6}$$

From λ_b, we can determine the error index b. The final step is to apply the operator F_b, where b is the value that we found for the error index.

$$F_b\left\{|\epsilon_b\rangle_{env.}F_b^\dagger|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\right\} = |\epsilon_b\rangle_{env.}|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.}\ . \tag{7}$$

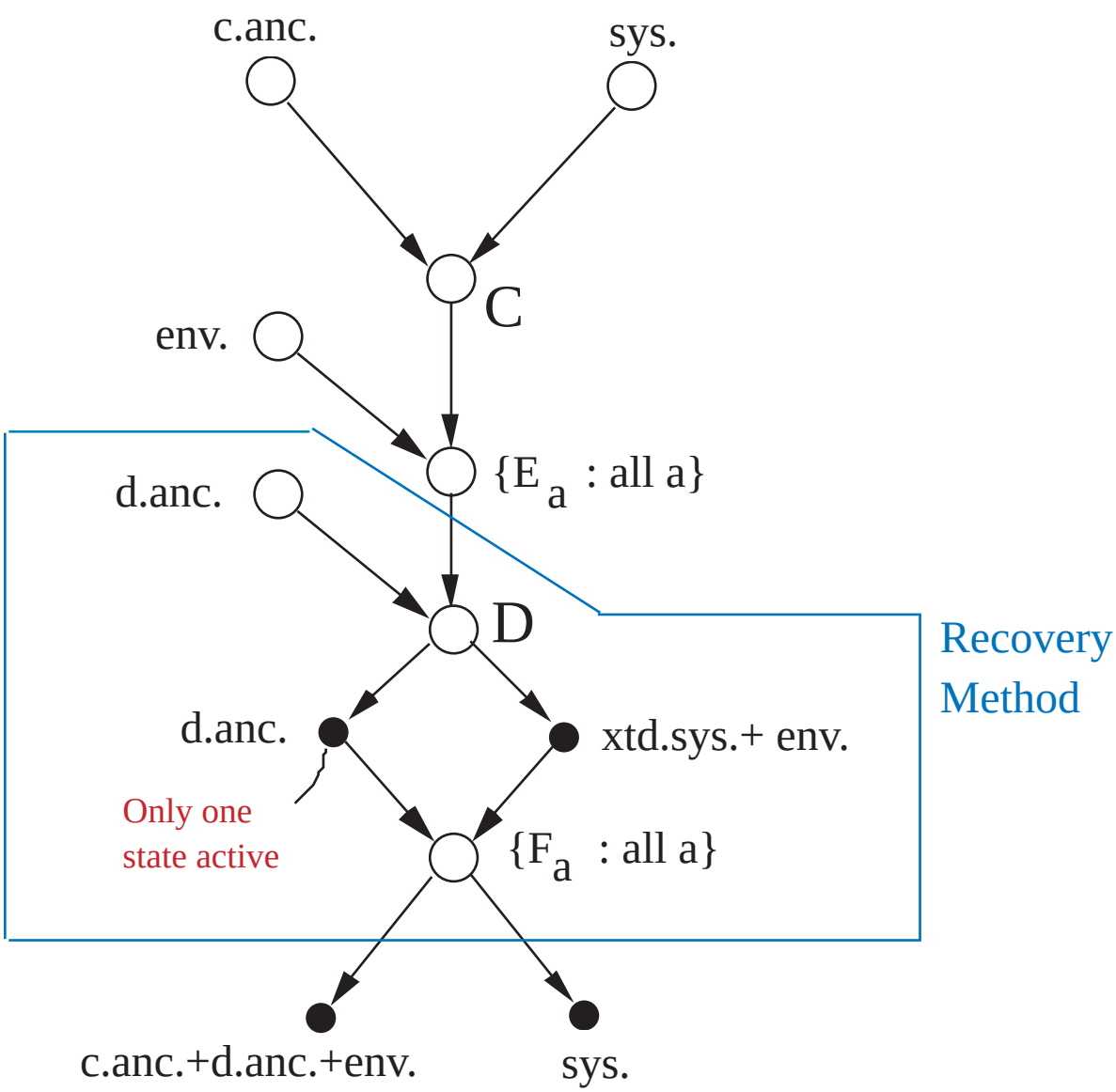

FIG. 3

Fig.3 gives a net representation of error correction using this recovery method. In this figure, the measurement of the diagnostic ancilla corresponds to making active only one state of the measured node.

Diagnosis Measurement Unnecessary

In Methods 2 and 3, an intermediate diagnosis measurement is performed. Actually, the nets of Figs.2 and 3 will still achieve the goal of error correction (i.e., to produce an output which is a tensor product of $|\phi_{in}\rangle_{sys.}$ times something else), even if no diagnosis measurement is performed (i.e., even if all node states are made active). We present a proof of this in a small appendix at the end of this section. We've seen something similar in the "Teleportation" and "Qbit Bouncing" sections. There we found that those effects can be accomplished with or without an intermediate measurement. Note that if no diagnosis measurement is performed, then the box marked "Recovery Method" in Fig.2 can be replaced by a single unitary operator called R. Hence, Fig.2 reduces to Fig.1. Likewise, Fig.3 reduces to a slightly modified version of Fig.1, the modification being the addition of a root node called *d.anc.* and an arrow pointing from this new root node to the node R.

Appendix: Proof That Diagnosis Measurement is Unnecessary

References

[1] S. Braunstein, "Quantum error correction of dephasing in 3 qubits", Los Alamos eprint quant-ph/9603024

[2] R. Laflamme, C. Miquel, Juan Pablo Paz, W. H. Zurek, *Phys. Rev. Let.* , **77**, 198 (1996). Also available as Los Alamos eprint quant-ph/9602019

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Appendix: Proof That Diagnosis Measurement is Unnecessary

In this appendix, we will present 2 proofs of the fact that error correction using Recovery Methods 2 and 3 does not require a diagnostic measurement. One proof will use the language of state vectors, the other the language of stories.

state vectors proof: Suppose bit 0 (the "system") starts in state $|\phi_{in}\rangle_{sys.}$, and everything else starts in state $|\xi_{in}\rangle_{other}$. Suppose (sys. + other) undergoes coding, interaction with the environment, and a recovery method with an intermediate diagnosis measurement. Suppose that after this, (sys. + other) is in state $|\xi_\alpha\rangle_{other}|\phi_{in}\rangle_{sys.}$. Then we can write

$$V\left[\frac{1}{\mathcal{N}_\alpha}\sum_\beta |\alpha\beta\rangle_{AB}\langle\alpha\beta|_{AB}\right] U|\xi_{in}\rangle_{other}|\phi_{in}\rangle_{sys.} = |\xi_\alpha\rangle_{other}|\phi_{in}\rangle_{sys.} \ . \tag{1}$$

Eq.(1) can be explained as follows. (sys. + other) undergoes some evolution that we represent by a unitary operator U. Then we separate (sys. + other) into two parts A and B, and we measure part A in some basis with eigenvalues α'. The measurement yields the value α. There is a basis for B with eigenvalues β. After the measurement of A, (sys. + other) undergoes further evolution which we represent by the unitary operator V. $\mathcal{N}_\alpha$ is just a normalization constant, a complex number that depends on α.

Multiplying both sides of Eq.(1) by $\mathcal{N}_\alpha$ and summing over all α yields

$$VU|\xi_{in}\rangle_{other}|\phi_{in}\rangle_{sys.} = \left[\sum_\alpha \mathcal{N}_\alpha|\xi_\alpha\rangle_{other}\right]|\phi_{in}\rangle_{sys.} \ . \tag{2}$$

The left side of Eq.(2) represents the situation in which (sys. + other) undergoes the same coding operation, interaction with the environment, and recovery method as it does in Eq.(1), except that this time the intermediate diagnosis measurement is not performed. As required for a successful error correction method, the right side of Eq.(2) is a tensor product of $|\phi_{in}\rangle_{sys.}$ times something else.

stories proof: Define $Z_N = \{1, 2, \ldots N\}$. Consider a net with N nodes labelled $\underline{x}_j$ where $j \in Z_N$. Let x_j be a state of node $\underline{x}_j$. If $1 \leq k_1 < k_2 < \ldots \leq N$ and if $S = \{k_1, k_2, \ldots\}$, define $(\underline{x}.)_S$ to be the vector $(\underline{x}_{k_1}, \underline{x}_{k_2}, \ldots)$, and define $(x.)_S$ to be the vector $(x_{k_1}, x_{k_2}, \ldots)$. (We also like to abbreviate $(\underline{x}.)_{Z_N}$ by $\underline{x}.$ and $(x.)_{Z_N}$ by $x.$).

Suppose that $A[\underline{x}. = x.]$ represents the amplitude of a story which assigns state x_j to node $\underline{x}_j$ for all $j \in Z_N$. Suppose Z_{int} gives the internal nodes of the net and Z_{ext} the external ones so that $Z_{int} \cup Z_{ext} = Z_N, Z_{int} \cap Z_{ext} = \emptyset$. Suppose $\underline{x}_s$ is the external node which represents the system at the end of the error correction process. Suppose $\underline{x}_d$ is the node that is measured to obtain a diagnosis. If the diagnosis measurement is performed, then only one state of node $\underline{x}_d$ is active. Call this state α. Then we can write for the nets of Figs.2 and 3:

$$\sum_{(x.)_{Z_{int}}} A[\underline{x}. = x.]\delta(x_d, \alpha) = g_\alpha[(x.)_{Z_{ext}-\{s\}}]f(x_s) \ , \tag{3}$$

where the sum on the left side is over all the possible values of $(x.)_{Z_{int}}$. In this equation, g_α is a function that depends on α and on the state of all the external nodes except $\underline{x}_s$. f is a function that depends only on the state x_s of node $\underline{x}_s$. Eq.(3) gives the wavefunction of the overall system when α is the only active state of node $\underline{x}_d$. The fact that the right side of Eq.(3) factors into a function that depends on x_s and another that doesn't is equivalent to the statement that the final state vector of the overall system is a tensor product of $|\phi_{in}\rangle_{sys.}$ times something else. Summing both sides of Eq.(3) over all α leads to

$$\sum_{(x.)_{Z_{int}}} A[\underline{x}. = x.] = \left\{ \sum_\alpha g_\alpha[(x.)_{Z_{ext}-\{s\}}] \right\} f(x_s) \ . \tag{4}$$

Eq.(4) gives the state vector of the overall system when all its nodes states are active. We see that it still factors into a function that depends on x_s and another that doesn't.

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Nets For Protecting A Single Qbit

In this section, we will present 2 examples of nets, Net1 and Net2, that perform error correction. Net1 and Net2 encode a single qbit (the "system") into 3 qbits. The 3 qbits interact with the environment in a way that may introduce an error of the following type: for Net1, flip of a single qbit; for Net2, dephasing of a single qbit. A final recovery process leaves the system in its original state and uncorrelated to everything else. Net1 and Net2 are both based on quantum circuits first suggested in Refs.[1]-[2].

Although we won't discuss it in this library, it is also possible to protect a single qbit against arbitrary errors of a single qbit. Indeed, Ref.[3] shows that it is possible to encode a single qbit into 5 qbits in such a way that recovery is possible, even if the 5 qbits interact with the environment in such a way that an arbitrary error of a single qbit can occur. The phrase "arbitrary error of a single qbit" means either a flip, a dephasing, or a flip and dephasing of a single qbit.

Consider first Net1.

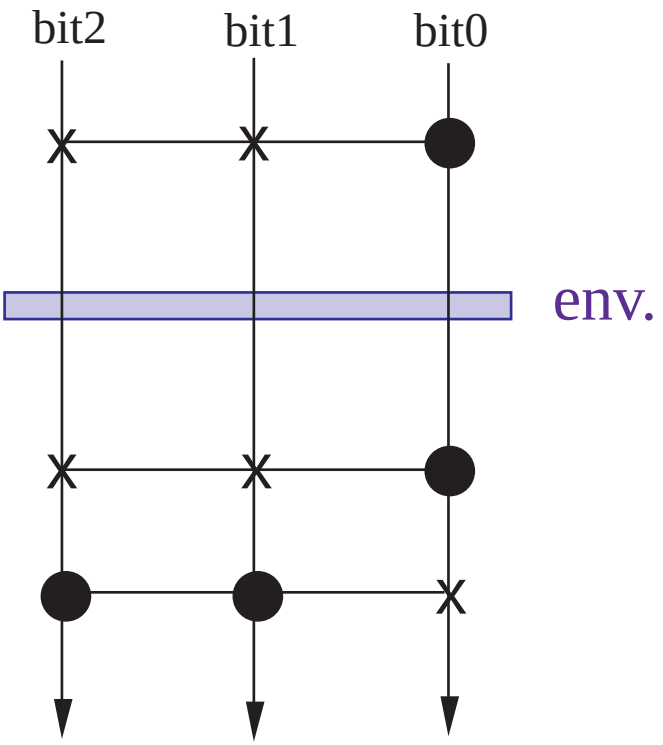

FIG. 1

Fig.1 shows the quantum circuit upon which Net1 is based.

Suppose that bit 0 is our system, and that it lies initially in state $|\phi_{in}\rangle_{sys.} = \alpha|0\rangle_0 + \beta|1\rangle_0$. Suppose bits 1 and 2 are our coding ancilla, and that they lie initially in state $|0\rangle_{c.anc.} = |0\rangle_2|0\rangle_1$.

Define operators C, E and R by

$$C = \sigma_{x2}^{n_0}\sigma_{x1}^{n_0} \ , \tag{1}$$

$$E = |\epsilon_u\rangle_{env.}1 + |\epsilon_0\rangle_{env.}\sigma_{x0} + |\epsilon_1\rangle_{env.}\sigma_{x1} + |\epsilon_2\rangle_{env.}\sigma_{x2} \ , \tag{2}$$

$$R = \sigma_{x0}^{n_1 n_2}\sigma_{x2}^{n_0}\sigma_{x1}^{n_0} \ . \tag{3}$$

C is the coding operator. E is an operator that can produce a single bit-flip error in (sys. + c.anc.). E also entangles (sys. + c.anc.) with the environment. R is the recovery operator.

One has

$$
\begin{aligned}
& REC|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.} \\
& = RE(\alpha|000\rangle + \beta|111\rangle) \\
& = R\left\{\begin{array}{rl} & |\epsilon_u\rangle(\alpha|000\rangle + \beta|111\rangle) \\ + & |\epsilon_0\rangle(\alpha|001\rangle + \beta|110\rangle) \\ + & |\epsilon_1\rangle(\alpha|010\rangle + \beta|101\rangle) \\ + & |\epsilon_2\rangle(\alpha|100\rangle + \beta|011\rangle) \end{array}\right\} . \\
& = \left\{\begin{array}{rl} & |\epsilon_u\rangle|00\rangle \\ + & |\epsilon_0\rangle|11\rangle \\ + & |\epsilon_1\rangle|01\rangle \\ + & |\epsilon_2\rangle|10\rangle \end{array}\right\}|\phi_{in}\rangle_{sys.}
\end{aligned}
\tag{4}
$$

Now open the Quantum Fog file entitled "Net1ForProt1Qbit".

- STATES: The root nodes $bit0$, $c.anc.$ and $env.$ have states $\{0,1\}$, $\{00,01,10,11\}$ and $\{ident, sigx0, sigx1, sigx2\}$, respectively. The states of node $env.$ correspond to what we called $|\epsilon_u\rangle, |\epsilon_0\rangle, |\epsilon_1\rangle, |\epsilon_2\rangle$ above.

  Node $CodingOp$ has 2 states, 000 and 111. This node represents the operator C above. One's first impulse is to give $CodingOp$ 8 states labelled by the binary numbers from 000 to 111. This would make node $CodingOp$ a unitary matrix. However, we soon realize that this would be an unnecessary complication because from Eq.(4) we can infer that all possible stories for the net (i.e., those with non-zero amplitude) will assign either state 000 or 111 to this node. The other 6 states, if included, would never be "visited".

  Nodes $Errors$ and $RecoveryOp$ both have 8 states. For these 2 nodes, the label of each state tells the individual state of each of three qbits and of the environment. Node $(c.anc. + env.)fin$ has 4 states. The label of each $(c.anc. + env.)fin$ state tells the individual state of each of the 2 coding ancilla and of the environment. Node $(bit0)fin$ has two states, 0 and 1.

- AMPLITUDES: We want the coding ancilla to lie initially in the state $|00\rangle$. Hence, we've assigned amplitude 1 to state 00 of node $c.anc.$ and 0 to all other states of that node. We have chosen some arbitrary amplitudes for the other 2 root nodes, $env.$ and $bit0$. The amplitudes of nodes $CodingOp, Errors$ and $RecoveryOp$ were inferred from Eq.(1) to Eq.(3) and inserted by hand. The amplitudes of nodes $(c.anc. + env.)fin$ and $(bit0)fin$ were both generated by pressing the **Generate Amplitudes Button...** of the **Node Prior-Info** window, and then selecting the Marginalizer dialog.

- INACTIVE STATES: Initially, all states of all nodes except node $env.$ are active, and only one state (it doesn't matter which one) of $env.$ is active. Afterwards, we make all node states, including those of node $env.$, active.

When only one state of node $env.$ is active, there are 2 possible endings, and one possible story with each ending. The 2 possible stories are in 1-1 correspondence

with the 2 possible joint states of the root nodes. (1 *c.anc.* state) times (1 *env.* state) times (2 $bit0$ states) = 2 joint states.

When all states of node *env.* are active, there are 8 possible endings, and one possible story with each ending. The 8 possible stories are in 1-1 correspondence with the 8 possible joint states of the root nodes. (1 *c.anc.* state) times (4 *env.* states) times (2 $bit0$ states) = 8 joint states.

When only one state of node *env.* is active, let $Cond.$ stand for the condition that *env.* equals that state. When all states of node *env.* are active, let $Cond.$ stand for nothing. Whether one or all states of *env.* are active, we get:

$$\begin{array}{l} P[bit0 = 0|Cond.] = P[(bit0)fin = 0|Cond.] \ , \\ P[bit0 = 1|Cond.] = P[(bit0)fin = 1|Cond.] \end{array} \quad . \tag{5}$$

Of course, Eq.(5) is not sufficient to prove that the state vectors of $bit0$ and $(bit0)fin$ are equal up to a phase factor. If these state vectors are $\alpha|0\rangle + \beta|1\rangle$ and $\alpha'|0\rangle + \beta'|1\rangle$, respectively, then Eq.(5) only establishes that

$$\begin{array}{l} |\alpha|^2 = |\alpha'|^2 \ , \\ |\beta|^2 = |\beta'|^2 \end{array} \quad . \tag{6a}$$

One must still prove that

$$\alpha\beta^* = \alpha'\beta'^* \ . \tag{6b}$$

Eq.(6b) could be established, for example, by adding a new qbit rotator, and drawing an arrow from $(bit0)fin$ to the new node. Measurements of the new node would yield further information about α' and β' which would permit us to prove conclusively that the state vectors of $bit0$ and $(bit0)fin$ were equal up to a phase factor.

Consider next Net2.

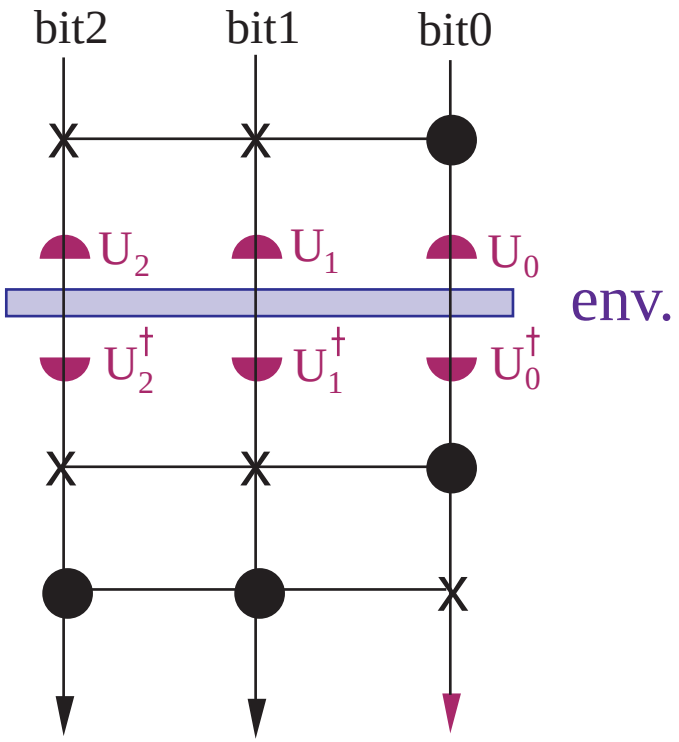

FIG. 2

Fig.2 shows the quantum circuit upon which Net2 is based.

Define U_β for $\beta \in \{0, 1, 2\}$ by

$$U_\beta = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}_\beta . \tag{7}$$

See the discussion about this transformation in the appendix entitled "Qbits". In particular, make sure that you understand how U_β rotates qbit β by 90 degrees about the Y axis, and, therefore, interchanges the X and Z Pauli matrices of qbit β. As in that appendix, we define the primed basis for qbit β by applying U_β to the standard basis of qbit β.

Let Γ be the product of the 3 rotations:

$$\Gamma = U_2 U_1 U_0 . \tag{8}$$

Define operators C', E' and R' in terms of Γ and the operators C, E and R that were defined by Eqs.(1) to (3):

$$C' = \Gamma C , \tag{9}$$

$$E' = \Gamma E \Gamma^\dagger = |\epsilon_u\rangle_{env.} 1 + |\epsilon_0\rangle_{env.} \sigma_{z0} + |\epsilon_1\rangle_{env.} \sigma_{z1} + |\epsilon_2\rangle_{env.} \sigma_{z2} \ , \tag{10}$$

$$R' = R\Gamma^\dagger \ . \tag{11}$$

Now a result analogous to Eq.(4) can be obtained. One gets

$$\begin{array}{l} R'E'C'|0\rangle_{c.anc.}|\phi_{in}\rangle_{sys.} \\ = R'E'(\alpha|0'0'0'\rangle + \beta|1'1'1'\rangle) \\ = R' \left\{ \begin{array}{lll} & |\epsilon_u\rangle(\alpha|0'0'0'\rangle + \beta|1'1'1'\rangle) \\ + & |\epsilon_0\rangle(\alpha|0'0'1'\rangle + \beta|1'1'0'\rangle) \\ + & |\epsilon_1\rangle(\alpha|0'1'0'\rangle + \beta|1'0'1'\rangle) \\ + & |\epsilon_2\rangle(\alpha|1'0'0'\rangle + \beta|0'1'1'\rangle) \end{array} \right\} \\ = \left\{ \begin{array}{lll} & |\epsilon_u\rangle|00\rangle \\ + & |\epsilon_0\rangle|11\rangle \\ + & |\epsilon_1\rangle|01\rangle \\ + & |\epsilon_2\rangle|10\rangle \end{array} \right\} |\phi_{in}\rangle_{sys.} \end{array} \ . \tag{12}$$

Open the Quantum Fog file entitled "Net2ForProt1Qbit".

Net2 is identical to Net1 except for some relabelling. Some of the states that were called 0 and 1 in Net1, are called $0'$ and $1'$, respectively, in Net2. Also, whereas the states of $env.$ were $\{ident, sigx0, sigx1, sigx2\}$ in Net1, they are $\{ident, sigz0, sigz1, sigz2\}$ in Net2. Except for this relabelling, most of what we said earlier about Net1 is also true about Net2.

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[1] Andrew Steane, *Proc.Roy.Soc.Lond.* **A452**, 2551 (1996). Also available as Los Alamos eprint quant-ph/9601029

[2] S. Braunstein, "Quantum error correction of dephasing in 3 qubits", Los Alamos eprint quant-ph/9603024

[3] R. Laflamme, C. Miquel, Juan Pablo Paz, W. H. Zurek, *Phys. Rev. Let.* , **77**, 198 (1996). Also available as Los Alamos eprint quant-ph/9602019

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Conditional Probabilities In Quantum Mechanics, What Do They Mean?

Quantum Mechanics (QM) predicts $P(X = x|Y = y)$, the probability that a measurement of X yields a value x, *conditioned on* the fact that (i.e., assuming that) a measurement of Y on the same system gave a value of y. For example, in Young's double slit experiment with a single photon, one can measure whether or not the photon went through a particular slit. If we do (ditto, do not) condition on this measurement, then QM predicts that interference will not (ditto, will) occur downstream.

A measurement is something done by the detectors and other boundary conditions inherent in the experimental setup. Measurements occur regardless of whether a particular observer is aware or unaware of them. Does this mean that in QM conditioning on $Y = y$ cannot be associated, like in classical probability, with the increase in knowledge of an observer? No. Such an association is still possible if you are careful: You can say that $Y = y$ represents the increase in knowledge of a *maximally alert observer*. $Y = y$ represents the maximum knowledge one can glean about the state of the system prior to the measurement of X.

How do we know what to condition on in a particular experimental situation? Well, suppose we compare our QM theoretical prediction for $P(X = x|Y = y)$ with our experimental result for the frequency with which we observe $X = x$. If they don't agree it means that the experimental setup is doing a measurement that we are unaware of. We figure what it is, say $Y' = y'$, and use QM to calculate $P(X = x|Y = y, Y' = y')$. We continue this process until our calculation matches the experimental result. It's not a fudge if the number of times you go back and revise your calculation is finite.

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Collapse Of State Vector, What Is It?

Consider first Classical Probability:

At any given time the system lies in a *fully-precise state*. The fully-precise state is observer independent. Given any property of the system, a fully-precise state is precise about what is the value of that property.

An observer pictures the system in an *imagined state* which depends on his imperfect knowledge about it. The imagined state is often unprecise about the value of some property of the system.

When the system is measured and an observer finds out the results of the measurement, his imagined state *collapses* into something more precise about the values of the properties of the system. The collapse of the imagined state of an observer occurs iff that observer acquires new knowledge.

Now consider Quantum Mechanics:

At any given time the system is in a *quantum state*. The quantum state is like an imagined state in that it is unprecise about the values of certain properties of the system. It is unlike an imagined state and more like the fully-precise state in that it does not depend on the particular observer.

When measured, the quantum state *collapses* into something more precise about the values of the properties of the system. Can one say that the collapse of the quantum state occurs iff an observer acquires new knowledge? Yes. But we can't say it for just any observer that might be asleep at the wheel. We can say it for a *maximally alert observer*. If we didn't ask for a maximally alert observer, since knowledge of the collapse could be different for different observers, the quantum state after the collapse would depend on the observer.

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Pauli Matrices

The Pauli matrices are

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{1}$$

They satisfy

$$\sigma_x \sigma_y = i\sigma_z , \quad \sigma_y \sigma_z = i\sigma_x , \quad \sigma_z \sigma_x = i\sigma_y , \tag{2}$$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 , \tag{3}$$

$$\sigma_x \sigma_y = -\sigma_y \sigma_x , \quad \sigma_y \sigma_z = -\sigma_z \sigma_y , \quad \sigma_z \sigma_x = -\sigma_x \sigma_z . \tag{4}$$

A useful identity for $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is

$$e^{i\vec{\theta}\cdot\vec{\sigma}} = \cos\theta + i\hat{\theta}\cdot\vec{\sigma}\sin\theta , \tag{5}$$

where

$$\theta = \sqrt{\theta_x^2 + \theta_y^2 + \theta_z^2} , \tag{6a}$$

$$\hat{\theta} = \frac{\vec{\theta}}{\theta} . \tag{6b}$$

Eq.(5) can be established in two steps: (1) Compare the Taylor series of both sides of the equation when $\vec{\theta}$ points in the positive X direction. (2) Rotate the coordinate system.

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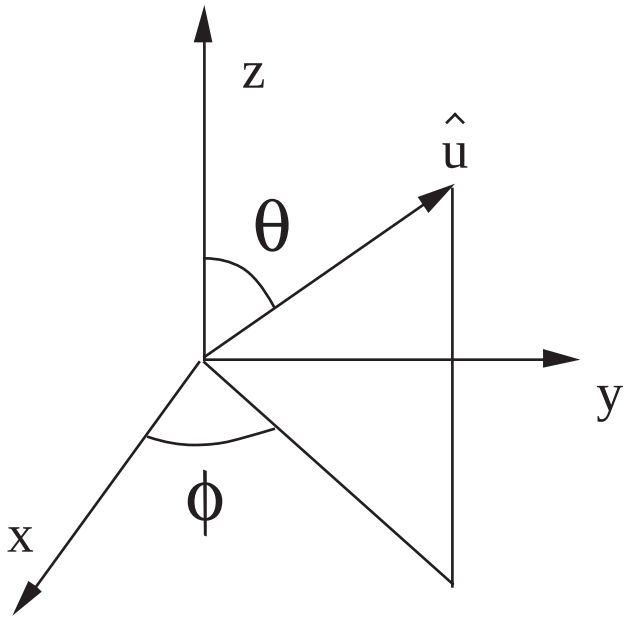
Spin 1/2 Particles



The following formalism is used to describe spin 1/2 particles.
Let $\hat{u}$ be a unit 3-dimensional vector characterized by angles (θ, ϕ) so that

$$\hat{u} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} . \tag{1}$$

Let $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ be the vector of Pauli matrices, where

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{2}$$

If $|+_u\rangle$ and $|-_u\rangle$ are defined by

$$\vec{\sigma}\cdot\hat{u}|+_u\rangle = |+_u\rangle , \quad \vec{\sigma}\cdot\hat{u}|-_u\rangle = -|-_u\rangle , \tag{3}$$

then one can show that

$$|+_u\rangle = \begin{bmatrix} CE^* \\ SE \end{bmatrix} , \quad |-_u\rangle = \begin{bmatrix} -SE^* \\ CE \end{bmatrix} , \tag{4}$$

where

$$S = \sin\frac{\theta}{2}\ ,\quad C = \cos\frac{\theta}{2}\ ,\quad E = \exp(i\frac{\phi}{2})\ . \tag{5}$$

For example, if $\theta = \phi = 0$, one gets

$$|+_z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\ ,\quad |-_z\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\ . \tag{6}$$

By Eqs.(4) and (5), if $\hat{u}$ and $\hat{u'}$ are unit vectors characterized by angles (θ_u, ϕ_u) and $(\theta_{u'}, \phi_{u'})$, respectively, and if $\phi_u = \phi_{u'} = 0$, then

$$\langle +_{u'}|+_u\rangle = \langle -_{u'}|-_u\rangle = \cos(\frac{\theta_{u'} - \theta_u}{2})\ , \tag{7}$$

$$\langle +_{u'}|-_u\rangle = -\langle -_{u'}|+_u\rangle = \sin(\frac{\theta_{u'} - \theta_u}{2})\ . \tag{8}$$

For any unit vector $\hat{u}$ and two spin 1/2 particles 1 and 2, the following state of particles 1 and 2 is of interest:

$$|\Psi_u^{ant}\rangle = \frac{1}{\sqrt{2}}\left[|+_u\rangle_1|-_u\rangle_2 - |-_u\rangle_1|+_u\rangle_2\right]\ . \tag{9}$$

$|\Psi_u^{ant}\rangle$ is antisymmetric (i.e., it changes sign if one interchanges the particle labels). By expressing the states $|\pm_u\rangle$ on the right side of Eq.(9) in terms of $|\pm_z\rangle$, it is easy to show that $|\Psi_u^{ant}\rangle$ is invariant under rotations, i.e., it is independent of $\hat{u}$. $|\Psi_u^{ant}\rangle$ has zero angular momentum. It is often called the singlet state, because it is the only possible zero angular momentum state for two spin 1/2 particles.

The following 3-particle state will arise in our studies:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|+_z\rangle_1|+_z\rangle_2|+_z\rangle_3 - |-_z\rangle_1|-_z\rangle_2|-_z\rangle_3]\ . \tag{10}$$

To deal with three spin 1/2 particles, one can use operators σ_x^i, σ_y^i and σ_z^i for $i \in \{1, 2, 3\}$. The index i labels the particle. Operators labelled by i act only on the state of particle i. Operators with different i's commute. Operators with the same i obey the same multiplication rules as the Pauli matrices of Eq.(2). Define operators Σ_1, Σ_2, and Σ_3 by

$$\Sigma_1 = \sigma_x^1\sigma_y^2\sigma_y^3\ ,\quad \Sigma_2 = \sigma_y^1\sigma_x^2\sigma_y^3\ ,\quad \Sigma_3 = \sigma_y^1\sigma_y^2\sigma_x^3\ . \tag{11}$$

Using the properties of Pauli matrices, it is easy to show that

$$\Sigma_1\Sigma_2\Sigma_3 = -\sigma_x^1\sigma_x^2\sigma_x^3\ . \tag{12}$$

One can also show that the state given by Eq.(10) satisfies

$$\Sigma_1|\Psi\rangle = +|\Psi\rangle\ ,\quad \Sigma_2|\Psi\rangle = +|\Psi\rangle\ ,\quad \Sigma_3|\Psi\rangle = +|\Psi\rangle\ . \tag{13}$$

Eqs.(12) and (13) imply that

$$\sigma_x^1 \sigma_x^2 \sigma_x^3 |\Psi\rangle = -\Sigma_1 \Sigma_2 \Sigma_3 |\Psi\rangle = -|\Psi\rangle \ . \tag{14}$$

So far, our formulation has been a first quantized one. In a second quantized formulation, one defines annihilation operators $a_{u\sigma}$, for any unit vector $\hat{u}$ and for $\sigma \in \{+, -\}$. These annihilation operators must satisfy the following anti-commutator relationships:

$$[a_{u\sigma}, a_{u'\sigma'}]_+ = 0 \ , \tag{15}$$

$$[a_{u\sigma}, a^\dagger_{u'\sigma'}]_+ = \delta(u, u')\delta(\sigma, \sigma') \ . \tag{16}$$

Antisymmetric states in the first quantized formulation are mapped in a 1-1 fashion into states in the second quantized formulation. For example,

$$|+_u\rangle \rightarrow a^\dagger_{u+}|0\rangle \ , \tag{17}$$

$$|-_u\rangle \rightarrow a^\dagger_{u-}|0\rangle \ , \tag{18}$$

$$|\Psi^{ant}\rangle \rightarrow a^\dagger_{u+} a^\dagger_{u-}|0\rangle \ . \tag{19}$$

From Eqs.(4), (17) and (18), it follows that

$$\begin{bmatrix} a^\dagger_{u+} \\ a^\dagger_{u-} \end{bmatrix} = \begin{bmatrix} CE^* & SE \\ -SE^* & CE \end{bmatrix} \begin{bmatrix} a^\dagger_{z+} \\ a^\dagger_{z-} \end{bmatrix} \ . \tag{20}$$

(To check this last equation, just apply $|0\rangle$ from the right to both sides of the equation.)

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Qbits

Qbits may be physical particles but they don't have to be. In this appendix, we present formalism which is commonly used when speaking of qbits in general. The appendix entitled "Spin 1/2 Particles" presents formalism which is commonly used when speaking of qbits that are particles.

We like to define $|0\rangle$ and $|1\rangle$ by

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \ \ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ . \tag{1}$$

If we follow this convention, then the Pauli matrix σ_x interchanges these two states ("flips the spin")

$$\sigma_x|0\rangle = |1\rangle \ , \ \ \sigma_x|1\rangle = |0\rangle \ , \tag{2}$$

whereas σ_z multiplies $|1\rangle$ by a phase factor, namely -1:

$$\sigma_z|0\rangle = |0\rangle \ , \ \ \sigma_z|1\rangle = -|1\rangle \ . \tag{3}$$

Since $\sigma_y = -i\sigma_z\sigma_x$, the effect of σ_y can be described as a spin flip followed by a change of phases.

In dealing with qbits, it is often convenient to use the "number operator" n, defined by

$$n = \frac{1-\sigma_z}{2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \ . \tag{4}$$

Note that n satisfies

$$n|0\rangle = 0|0\rangle \ , \ \ n|1\rangle = 1|1\rangle \ . \tag{5}$$

Suppose the state vector $|\phi\rangle$ of a qbit is represented by a 2-component column vector. When the qbit is rotated by an angle α about the $\hat{\alpha}$ direction, its state vector becomes $R_{\hat{\alpha}}(\alpha)|\phi\rangle$, where

$$R_{\hat{\alpha}}(\alpha) = e^{i\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} \ . \tag{6}$$

Note the factor of 1/2 in the argument of the exponential.

A particular rotation that is often useful is:

$$U = R_y(\frac{\pi}{2}) = e^{i\frac{\pi}{4}\sigma_y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} . \tag{7}$$

Note that U interchanges the X and Z Pauli matrices. Thus, it interchanges bit-flipping with bit-dephasing:

$$U^\dagger \sigma_z U = \sigma_x \ , \ \ U\sigma_z U^\dagger = -\sigma_x \ . \tag{8}$$

It is often useful to define a primed basis by applying U to the standard basis given by Eq.(1):

$$|0'\rangle = U|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \ , \ \ |1'\rangle = U|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{9}$$

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Beam-Splitter For Bosons

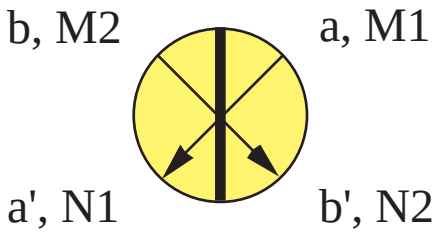

FIG.1

Out of all possible bosons, we will consider here only the special case of photons, which are spin-one massless bosons.

A photonic beam-splitter has two input and two output beams of photons. Let a and b represent the complex amplitudes of the two incoming beams, and a' and b' those of the outgoing beams. The following commutator relationships must be satisfied:

$$[a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, b] = [a, b^\dagger] = 0 \; . \tag{1}$$

Furthermore, Eq.(1) must be satisfied with a replaced by a' and b replaced by b'. One describes this last requirement by saying that the commutators must be preserved.

Suppose that

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = U \begin{bmatrix} a \\ b \end{bmatrix} \; , \tag{2}$$

where U is a 2×2 matrix. A necessary and sufficient condition for the commutators to be preserved is that U be a unitary matrix. Therefore, U must be of the form:

$$U = \begin{bmatrix} t & r \\ -r^* & t^* \end{bmatrix} \; , \tag{3}$$

where

$$|t|^2 + |r|^2 = 1 \; . \tag{4}$$

t and r are called the complex transmission and reflection coefficients of the beam splitter.

Let M_1, M_2, N_1 and N_2 be the number of photons in the a, b, a' and b' modes, respectively (See Fig.1). Then the amplitude Amp of the beam-splitter is

$$\text{Amp} = \langle 0| \frac{(a')^{N_1}}{\sqrt{N_1!}} \frac{(b')^{N_2}}{\sqrt{N_2!}} \frac{(a^\dagger)^{M_1}}{\sqrt{M_1!}} \frac{(b^\dagger)^{M_2}}{\sqrt{M_2!}} |0\rangle \ . \tag{5}$$

The previous equations easily yield:

$$\text{Amp} = \alpha \delta(N_1 + N_2, M_1 + M_2) \sqrt{N_1! N_2! M_1! M_2!} \ , \tag{6a}$$

where

$$\alpha = \sum_{j=max(0,M_1-N_2)}^{min(N_1,M_1)} \frac{t^j}{j!} \frac{(t^*)^{N_2-M_1+j}}{(N_2 - M_1 + j)!} \frac{r^{N_1-j}}{(N_1 - j)!} \frac{(-r^*)^{M_1-j}}{(M_1 - j)!} \ . \tag{6b}$$

Quantum Fog uses Eqs.(6) to calculate Amp.

The values of Amp for one incoming photon are as follows:

One Photon Amplitudes :

	$M_2 = 0, M_1 = 1$	$M_2 = 1, M_1 = 0$
$N_1 = 1, N_2 = 0$	t	r
$N_1 = 0, N_2 = 1$	$-r^*$	t^*

The one photon amplitudes may be represented graphically as follows:

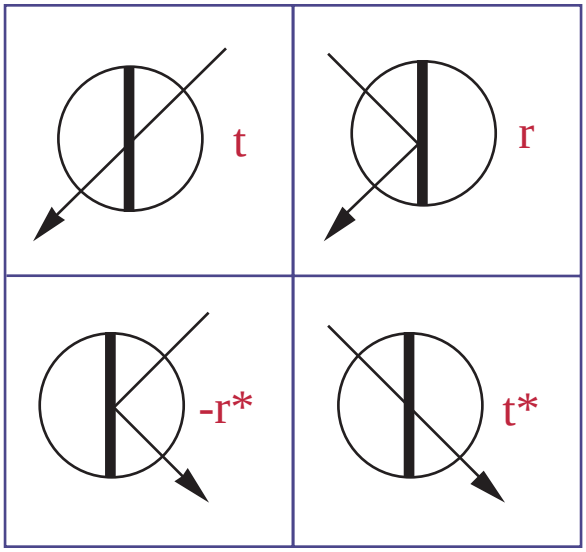

FIG.2

The values of Amp for two incoming photons are as follows:

Two Photon Amplitudes :

	$M_2 = 0, M_1 = 2$	$M_2 = 1, M_1 = 1$	$M_2 = 2, M_1 = 0$
$N_1 = 2, N_2 = 0$	t^2	$tr\sqrt{2}$	r^2
$N_1 = 1, N_2 = 1$	$t(-r^*)\sqrt{2}$	$\|t\|^2 - \|r\|^2$	$t^* r\sqrt{2}$
$N_1 = 0, N_2 = 2$	$(-r^*)^2$	$t^*(-r^*)\sqrt{2}$	$(t^*)^2$

The above results are only valid if the inputs to the beam-splitter both have the same polarization direction. It's also possible to consider a beam-splitter for which the 2 inputs don't have the same polarization. In such a case, each of the modes a, b, a', b' is replaced by a 2-component vector. For example, a is replaced by $\vec{a} = (a_x, a_y)$, where a_x (ditto, a_y) represents the amplitude of photons polarized in the X (ditto, Y) direction. The operators dealing with the X polarization commute with those dealing with the Y polarization. Also, $[a_x, a_x^\dagger] = 1$ and $[a_y, a_y^\dagger] = 1$. Furthermore, each of the integers N_1, N_2, M_1, M_2 is replaced by a 2-component vector. For example, M_1 is replaced by $\vec{M}_1 = (M_{1x}, M_{1y})$, where M_{1x} (ditto, M_{1y}) represents the number of photons polarized in the X (ditto, Y) direction. If A_V is the amplitude of the beam-splitter when it has vector inputs, and A_S is the amplitude with scalar inputs, then

$$A_V(\vec{N}_1, \vec{N}_2, \vec{M}_1, \vec{M}_2) = A_S(N_{1x}, N_{2x}, M_{1x}, M_{2x}) A_S(N_{1y}, N_{2y}, M_{1y}, M_{2y}) \ , \qquad (7)$$

where A_S is given by Eqs.(6). Note that we are taking the parameters t and r to be the same for both polarizations. Quantum Fog uses Eq.(7) to calculate the amplitudes of a beam-splitter with vector inputs.

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Beam-Splitter For Fermions

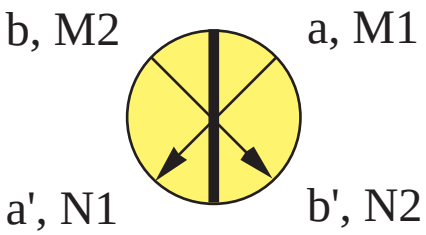

FIG.1

Out of all possible fermions, we will consider here only the special case of uncharged spin 1/2 fermions. A treatment of charged fermions would have to consider their electro-magnetic interaction. Neutrons are a familiar example of uncharged spin 1/2 fermions.

A fermionic beam-splitter has two input and two output beams of fermions. Let a and b represent the complex amplitudes of the two incoming beams, and a' and b' those of the outgoing beams. The following anti-commutator relationships must be satisfied:

$$[a, a^\dagger]_+ = [b, b^\dagger]_+ = 1, \quad [a, b]_+ = [a, b^\dagger]_+ = 0 \; . \tag{1}$$

Furthermore, Eq.(1) must be satisfied with a replaced by a' and b replaced by b'. One describes this last requirement by saying that the anti-commutators must be preserved.

Suppose that

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = U \begin{bmatrix} a \\ b \end{bmatrix} \; , \tag{2}$$

where U is a 2×2 matrix. A necessary and sufficient condition for the anti-commutators to be preserved is that U be a unitary matrix. Therefore, U must be of the form:

$$U = \begin{bmatrix} t & r \\ -r^* & t^* \end{bmatrix} \; , \tag{3}$$

where

$$|t|^2 + |r|^2 = 1 \; . \tag{4}$$

t and r are called the complex transmission and reflection coefficients of the beam-splitter.

Let M_1, M_2, N_1 and N_2 be the number of fermions in the a, b, a' and b' modes, respectively (See Fig.1). Because of the anti-commutators given in Eq.(1), M_1, M_2, N_1 and N_2 can only be either 0 or 1. In other words, fermionic modes can only be occupied by at most one particle. This agrees with the so called Pauli Exclusion Principle. The amplitude Amp of the fermionic beam-splitter is

$$\text{Amp} = \langle 0| \frac{(a')^{N_1}}{\sqrt{N_1!}} \frac{(b')^{N_2}}{\sqrt{N_2!}} \frac{(a^\dagger)^{M_1}}{\sqrt{M_1!}} \frac{(b^\dagger)^{M_2}}{\sqrt{M_2!}} |0\rangle \ . \tag{5}$$

If in Eq.(5) we express the primed modes in terms of the unprimed ones, we find that:

- If zero or one fermion enters the beam-splitter, then the amplitude of a fermionic beam-splitter is identical to that of a bosonic beam-splitter.
- If two or more fermions enter the beam-splitter, then Amp is zero unless $M_1 = M_2 = N_1 = N_2 = 1$. In the latter case, Amp equals -1.

The above results are only valid if the inputs to the beam-splitter both have the same spin direction (i.e., they both have spin up along some direction, call it Z.) It's also possible to consider a beam-splitter for which the 2 inputs don't have the same spin direction. In such a case, each of the modes a, b, a', b' is replaced by a 2-component vector. For example, a is replaced by $\vec{a} = (a_-, a_+)$, where a_- (ditto, a_+) represents the amplitude of fermions with spin down (ditto, up) along the Z direction. The operators dealing with spin down anti-commute with those dealing with spin up. Also, $[a_-, a_-^\dagger]_+ = 1$ and $[a_+, a_+^\dagger]_+ = 1$. Furthermore, each of the integers N_1, N_2, M_1, M_2 is replaced by a 2-component vector. For example, M_1 is replaced by $\vec{M}_1 = (M_{1-}, M_{1+})$, where M_{1-} (ditto, M_{1+}) represents the number of particles with spin down (ditto, up). If A_V is the amplitude of the beam-splitter when it has vector inputs, and A_S is the amplitude with scalar inputs, then

$$A_V(\vec{N}_1, \vec{N}_2, \vec{M}_1, \vec{M}_2) = A_S(N_{1-}, N_{2-}, M_{1-}, M_{2-}) A_S(N_{1+}, N_{2+}, M_{1+}, M_{2+}) \ , \tag{6}$$

where A_S is given by Eqs.(5). Note that we are taking the parameters t and r to be the same for both spin up and down.

Quantum Fog does not have a special node called a fermionic beam-splitter. You may, however, generate the amplitudes for such a node in the following way. Use a regular beam-splitter node. Assume that both parent nodes have precisely two states, called 0 and 1, representing the number of spin 1/2 particles with spin up along the Z direction that pass through these nodes. Press the **Generate Amplitudes...** button of the **Node Prior-Info.** window. Quantum Fog will generate the amplitudes assuming scalar photons. These amplitudes will apply to the fermions whenever zero or one particle enters the beam-splitter. Set to zero all amplitudes for more than

one input particle, and then set to -1 the amplitude for $M_1 = M_2 = N_1 = N_2 = 1$. You should also remove the states $(2,0),(0,2)$, since they have zero amplitude for any of the possible input states. (To remove $(2,0),(0,2)$, reorder the states of the beam-splitter so that these 2 states are last, and then reduce the number of rows by 2.)

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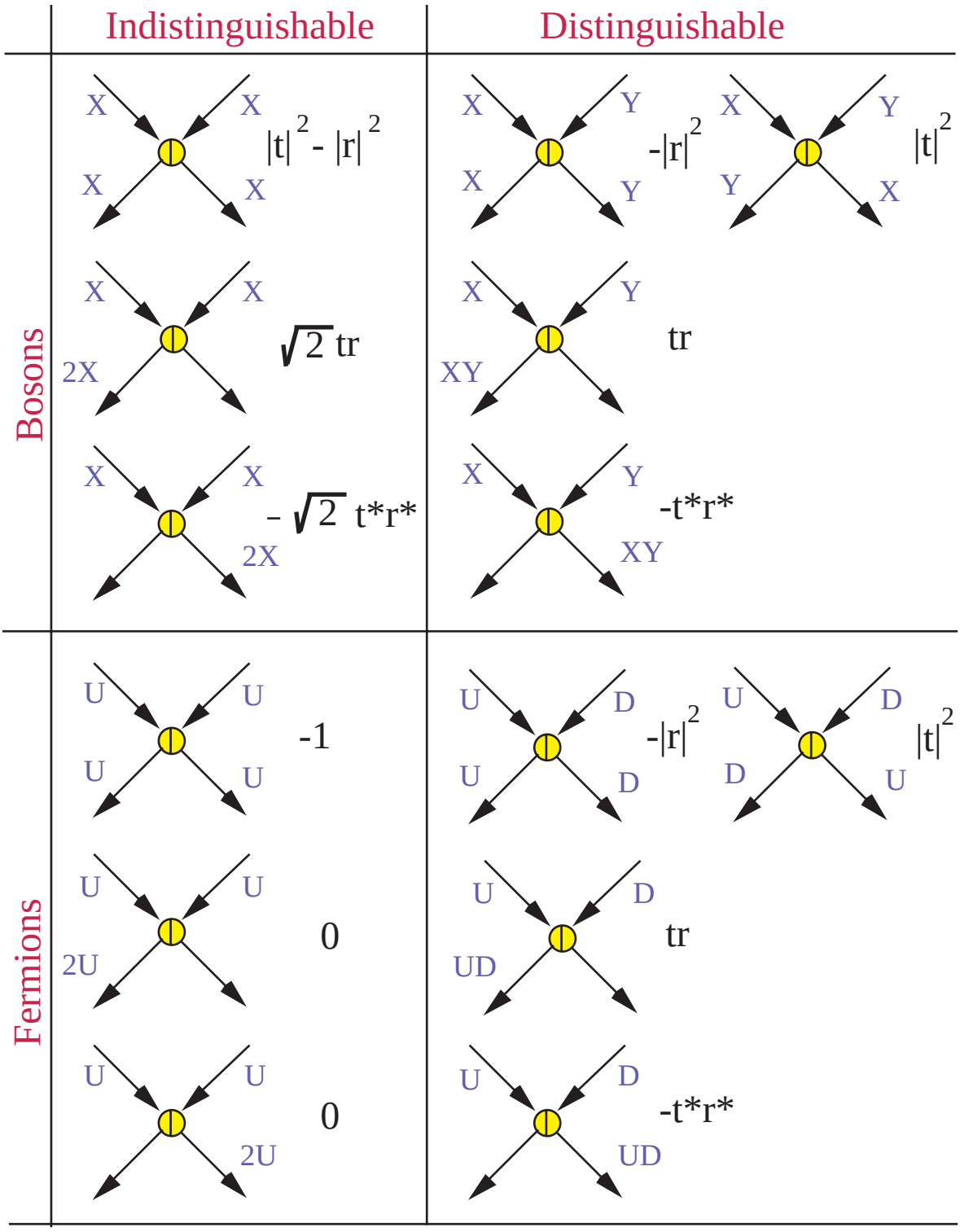
Beam-Splitter And Spin Statistics

Consider a net consisting of a single beam-splitter and 4 deterministic nodes, one attached to each of the 4 ports of the beam-splitter. Suppose that a particle enters each of the 2 input ports of the beam-splitter, and that these 2 particles meet inside the beam-splitter. Four cases are of interest to us, depending on whether the 2 input particles are 2 spin 1/2 fermions or 2 photons (spin-one bosons), and whether the particles are indistinguishable inside the beam-splitter or not. It is highly instructive to compare the possible stories for these 4 cases.

In the figure below, X and Y stand for the X and Y polarization directions of photons. U and D stand for the up and down spin projections of spin 1/2 fermions. Fermions with the same (ditto, different) spin projection are indistinguishable (ditto, distinguishable) inside the beam-splitter. Bosons with the same (ditto, different) polarization direction are indistinguishable (ditto, distinguishable) inside the beam-splitter.



The above figure teaches us some important lessons.

- When particles become indistinguishable inside a beam-splitter, and the beam-

splitter provides a symmetric environment ($|t| = |r| = 1/\sqrt{2}$), the bosons attract each other and leave by the same output port. Fermions, on the other hand, repel each other and leave by different output ports, in harmony with the so called Pauli Exclusion Principle. If the environment is not symmetric, then the previously identified attraction between bosons is "not strong enough" to make them exit through the same output port. But the previously identified repulsion of fermions is still strong enough to make them leave by different output ports.

- The behavior of a net as some of its particles go from being distinguishable to being indistinguishable can be highly discontinuous. This is true regardless of whether the particles making the transition are bosons or fermions. For example, look at the first row for bosons (ditto, first row for fermions). For bosons (ditto, fermions), 2 stories, call them S1 and S2, with different endings are replaced by a single story whose amplitude is the sum (ditto, difference) of the amplitudes of S1 and S2.

The above results assume mono-energetic particles that meet inside the beam-splitter. See Ref.[1] for an analysis of the case where the particles have finite energy spread and reach the beam-splitter at possibly different times.

References

[1] R. Loudon, Rochester Conference, *Coherence and Quantum Optics* 6, June 1989.

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Controlled-Not

You should read the appendix entitled "Qbits" before reading this appendix.

Suppose that α and β are two qbits with basis states $|M_\alpha, M_\beta\rangle_{\alpha\beta}$, where $M_\alpha, M_\beta \in \{0, 1\}$. Let $\sigma_{x\mu}$ and n_μ be the X Pauli matrix and the number operator for bit $\mu \in \{\alpha, \beta\}$. Consider the effect of the operator $\sigma_{x\beta}^{n_\alpha}$ on $|M_\alpha, M_\beta\rangle_{\alpha\beta}$

$$\sigma_{x\beta}^{n_\alpha}|00\rangle = \sigma_{x\beta}^{0}|00\rangle = |00\rangle \ , \tag{1a}$$

$$\sigma_{x\beta}^{n_\alpha}|01\rangle = \sigma_{x\beta}^{0}|01\rangle = |01\rangle \ , \tag{1b}$$

$$\sigma_{x\beta}^{n_\alpha}|10\rangle = \sigma_{x\beta}^{1}|10\rangle = |11\rangle \ , \tag{1c}$$

$$\sigma_{x\beta}^{n_\alpha}|11\rangle = \sigma_{x\beta}^{1}|11\rangle = |10\rangle \ . \tag{1d}$$

In the case of operator $\sigma_{x\beta}^{n_\alpha}$, we call α the *control* bit and β the *flipper* bit, and we say that the flipper bit flips whenever the state of the control bit is 1. A related operator is $\sigma_{x\beta}^{1-n_\alpha}$, which has the same control and flipper bits as the previous operator, but for which the flipper flips when the state of the control is 0. Two other related operators are $\sigma_{x\alpha}^{n_\beta}$ and $\sigma_{x\alpha}^{1-n_\beta}$, which differ from the previous two in that the flipper and control bits have been interchanged. Call *Ops* the set of the four operators just mentioned:

$$Ops = \{\sigma_{x\beta}^{n_\alpha}, \sigma_{x\beta}^{1-n_\alpha}, \sigma_{x\alpha}^{n_\beta}, \sigma_{x\alpha}^{1-n_\beta}\} \ . \tag{2}$$

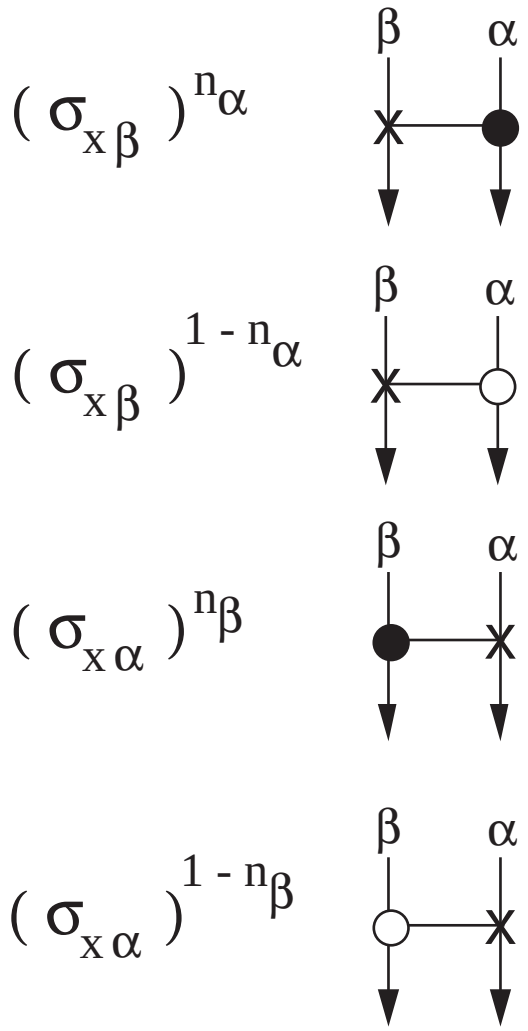

FIG. 1

Fig.1 shows the pictorial representation which is used in the literature for each of these four operators. In these diagrams, flipper bits are marked by an X and control bits by either a filled or an empty circle. A filled (ditto, empty) circle indicates that the flipper will flip when the state of the control is 1 (ditto, 0).

In Quantum Fog, we refer to all the nodes of Fig.1 as controlled-nots. Suppose a controlled-not node is entered by two bits, α and β, in states $M_\alpha, M_\beta \in \{0, 1\}$. Suppose the node is in a state (N, N'), where $N, N' \in \{0, 1\}$. The amplitudes $A(N, N'|M_\alpha, M_\beta)$ for the controlled-not node are given by the equations:

$$\Omega|M_\alpha, M_\beta\rangle = \sum_{N=0}^{1} \sum_{N'=0}^{1} |N, N'\rangle A(N, N'|M_\alpha, M_\beta) \ , \tag{3}$$

for $M_\alpha, M_\beta \in \{0, 1\}$. Ω is the particular operator in the set *Ops* which the controlled-not node represents. In the case that $\Omega = \sigma_{x\beta}^{n_\alpha}$, these four equations are just Eqs.(1) above. Quantum Fog uses Eqs.(3) to calculate the amplitudes of a controlled-not node.

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Loss Node

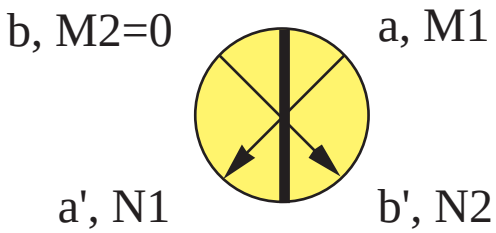

FIG.1

Classically, passage of a photon beam though a lossy medium transforms the complex amplitude of the beam from a to a' so that

$$a' = ta \ , \tag{1}$$

where t is a complex number that satisfies $|t| \leq 1$.

In Quantum Mechanics, the following commutator relationship must be satisfied: $[a, a^\dagger] = 1$. Furthermore, this relationship must be satisfied with a replaced by a', a requirement often described by saying that the commutator must be preserved. Eq.(1) above does not preserve the commutator. One can remedy this situation by adding a "loss mode" b to the transformation. Suppose b has zero mean value ($< b >= 0$), and

$$[a, a^\dagger] = [b, b^\dagger] = 1, \quad [a, b] = [a, b^\dagger] = 0 \ . \tag{2}$$

Suppose U is a 2×2 matrix such that

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = U \begin{bmatrix} a \\ b \end{bmatrix} \ . \tag{3}$$

We would like U to be unitary because that would imply that the commutators Eqs.(2) are preserved. We would also like U to be such that when one takes the mean value of both sides of Eq.(3) and uses $< b >= 0$, one obtains the mean value of Eq.(1). The following matrix satisfies these 2 conditions:

$$U = \begin{bmatrix} t & r \\ -r^* & t^* \end{bmatrix} \ , \tag{4}$$

where $|t|^2 + |r|^2 = 1$.

Eqs.(3) and (4) are identical to the transformation that describes a beam-splitter. It follows that a good model for lossy transmission is a beam-splitter (see Fig.1) for which M_2 is zero, and N_2 is the number of photons that are absorbed. We say that darkness or vacuum is entering the b port. According to the appendix entitled "Beam-Splitter", the amplitude for such a beam-splitter is:

$$\text{Amp} = \delta(N_1 + N_2, M_1)\sqrt{\frac{(N_1 + N_2)!}{N_1!N_2!}}t^{N_1}(-r^*)^{N_2} \ . \tag{5}$$

Quantum Fog does not have a special node called a loss node. To implement a loss node, use a beam-splitter node with $M2 = 0$. Interpret the b and b' ports in the way discussed above. With such a beam-splitter for a focus node, press the **Generate Amplitudes...** button of the **Node Prior-Info.** window. After filling out the ensuing dialog and pressing OK, Quantum Fog will use Eq.(5) to evaluate the amplitudes of the node.

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Marginalizer

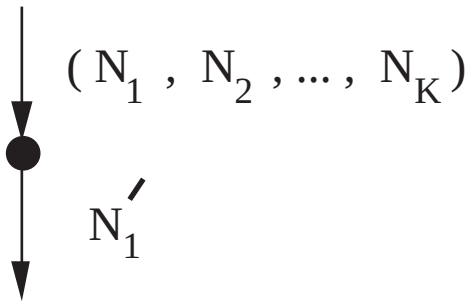

FIG.1

Suppose a node has a single parent, and the states of that parent are labelled by K-tuples $(N_1, N_1, \ldots, N_K)$, where $N_j \in S_j$ for all $j \in \{1, 2, \cdots, K\}$. For each j, S_j might be a subset of the integers but it need not be. A marginalizer node marginalizes (projects out) one of the K components of its parent node. For definiteness, let's say it projects out the first of these. Then the states of the marginalizer node are the elements of set S_1, and the amplitudes of the node are

$$A(N_1'|N_1, N_2, \ldots, N_K) = \delta(N_1', N_1) \ . \tag{1}$$

Quantum Fog uses Eq.(1) to calculate the amplitudes of a marginalizer.

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Phase-Shifter

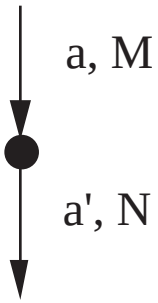

FIG.1

When a phase-shifter acts on a photon beam, it transforms the complex amplitude of the beam from a to a' so that

$$a' = e^{i\theta} a \ , \tag{1}$$

where θ is some real number.

In Quantum Mechanics, the following commutator relationship must satisfied: $[a, a^\dagger] = 1$. Let M and N be the number of photons in the modes a and a', respectively. Then the amplitude Amp of the phase-shifter is

$$\text{Amp} = \langle 0| \frac{(a')^N}{\sqrt{N!}} \frac{(a^\dagger)^M}{\sqrt{M!}} |0\rangle = \delta(N, M) e^{iN\theta} \ . \tag{2}$$

Quantum Fog uses Eq.(2) to calculate the amplitudes of a phase-shifter.

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Polarization Rotator

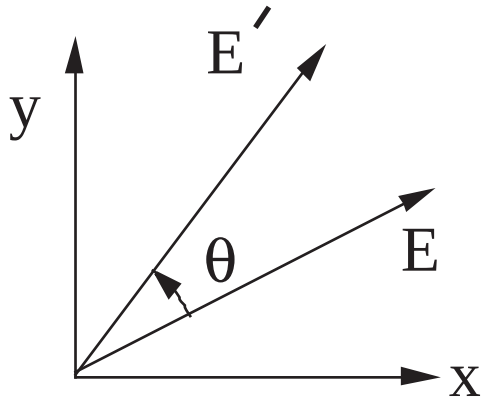

FIG.1

A polarization rotator transforms the electric field of a light beam from $\vec{E} = (a_x, a_y)$ to $\vec{E}' = (a'_x, a'_y)$ so that

$$\begin{bmatrix} a'_x \\ a'_y \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} . \tag{1}$$

In this equation, $C = \cos\theta$ and $S = \sin\theta$, where θ is the rotation angle. Eq.(1) is identical to a beam-splitter (with scalar-field inputs) transformation, except that in the beam-splitter case the 2 input modes have different propagation directions and the same polarization direction, whereas now the 2 input modes have the same propagation direction and different polarization directions. Therefore, a formula for evaluating the amplitudes of a polarization rotator can be obtained trivially from the formula used to evaluate the amplitudes of a beam-splitter.

Let M_x, M_y, N_x and N_y represent the number of photons in the modes a_x, a_y, a'_x and a'_y, respectively. Then we get from the appendix entitled "Beam-Splitter" that

$$\text{Amp} = \alpha\delta(N_x + N_y, M_x + M_y)\sqrt{N_x!N_y!M_x!M_y!} , \tag{2a}$$

where

$$\alpha = \sum_{j=max(0,M_x-N_y)}^{min(N_x,M_x)} \frac{C^j}{j!} \frac{C^{N_y-M_x+j}}{(N_y-M_x+j)!} \frac{S^{N_x-j}}{(N_x-j)!} \frac{(-S)^{M_x-j}}{(M_x-j)!} . \tag{2b}$$

Quantum Fog uses this last equation to calculate Amp.

The values of Amp for one incoming photon are as follows:

One Photon Amplitudes :

	$M_x = 1, M_y = 0$	$M_x = 0, M_y = 1$
$N_x = 1, N_y = 0$	C	S
$N_x = 0, N_y = 1$	$-S$	C

The one photon amplitudes may be represented graphically as follows:

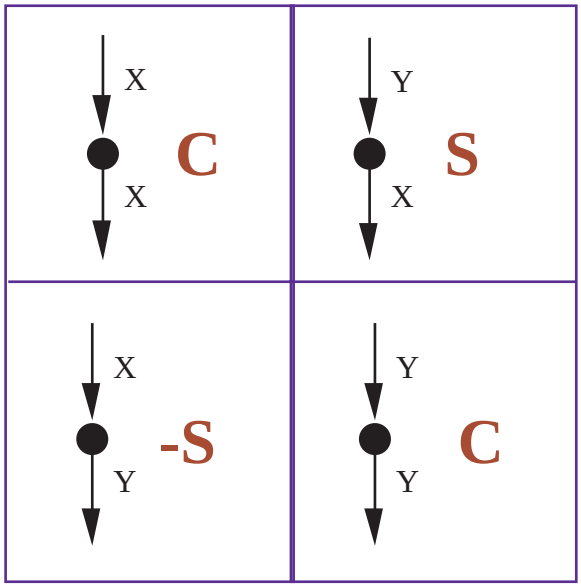

FIG.2

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Polarizer

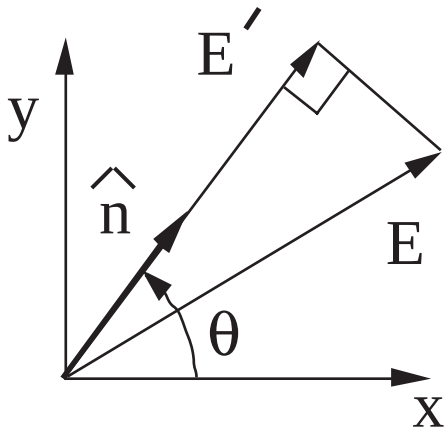

FIG.1

Classically, a polarizer transforms the electric field of a light beam from $\vec{E}$ to $\vec{E}'$ so that

$$\vec{E}' = \hat{n}(\hat{n} \cdot \vec{E}) \, , \tag{1}$$

where $\hat{n}$ is a unit vector directed along the polarization axis. Let

$$\vec{E} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} \;\; , \;\; \vec{E}' = \begin{bmatrix} a'_x \\ a'_y \end{bmatrix} \;\; , \;\; \hat{n} = \begin{bmatrix} C \\ S \end{bmatrix} \, . \tag{2}$$

In the last equation, $C = \cos\theta$ and $S = \sin\theta$, where θ is the angle that $\hat{n}$ makes with the X axis. Using Eqs.(2), Eq.(1) can be rewritten as

$$\begin{bmatrix} a'_x \\ a'_y \end{bmatrix} = \begin{bmatrix} C \\ S \end{bmatrix} \begin{bmatrix} C & S \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} \, . \tag{3}$$

In Quantum Mechanics, the following commutator relationships must be satisfied:

$$[a_\alpha, a^\dagger_\beta] = \delta(\alpha, \beta) \;\; , \;\; [a_\alpha, a_\beta] = 0 \, , \tag{4}$$

for all $\alpha, \beta \in \{x, y\}$. Furthermore, Eq.(4) must be satisfied with a_α replaced by a'_α and a_β replaced by a'_β. One describes this last requirement by saying that the

commutators must be preserved. Eq.(3) above does not preserve the commutators. One can remedy this situation by adding a "loss mode" b to the transformation. Suppose b has zero mean value ($< b >= 0$), and Eq.(4) is valid for $\alpha, \beta \in \{x, y, loss\}$, where $a_{loss} = b$. Suppose U is a 3×3 matrix such that

$$\begin{bmatrix} a'_x \\ a'_y \\ b' \end{bmatrix} = U \begin{bmatrix} a_x \\ a_y \\ b \end{bmatrix} . \tag{5}$$

We would like U to be unitary because that would imply that the commutators are preserved. We would also like U to be such that when one takes the mean value of both sides of Eq.(5) and uses $< b >= 0$, one obtains the mean value of Eq.(3). The following matrix satisfies these 2 conditions:

$$U = \begin{bmatrix} C^2 & CS & -S \\ CS & S^2 & C \\ -S & C & 0 \end{bmatrix} . \tag{6}$$

Not only is U unitary. It is in fact a 3-D rotation, and it can be expressed as a product of two 2-D rotations:

$$U = R_1 R_2 , \tag{7}$$

where

$$R_1 = \begin{bmatrix} C & 0 & -S \\ S & 0 & C \\ 0 & 1 & 0 \end{bmatrix} , \tag{8a}$$

$$R_2 = \begin{bmatrix} C & S & 0 \\ -S & C & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{8b}$$

Hence, the transformation given by Eq.(5) is equivalent to the successive application of 2 beam-splitter transformations (See Fig.2).

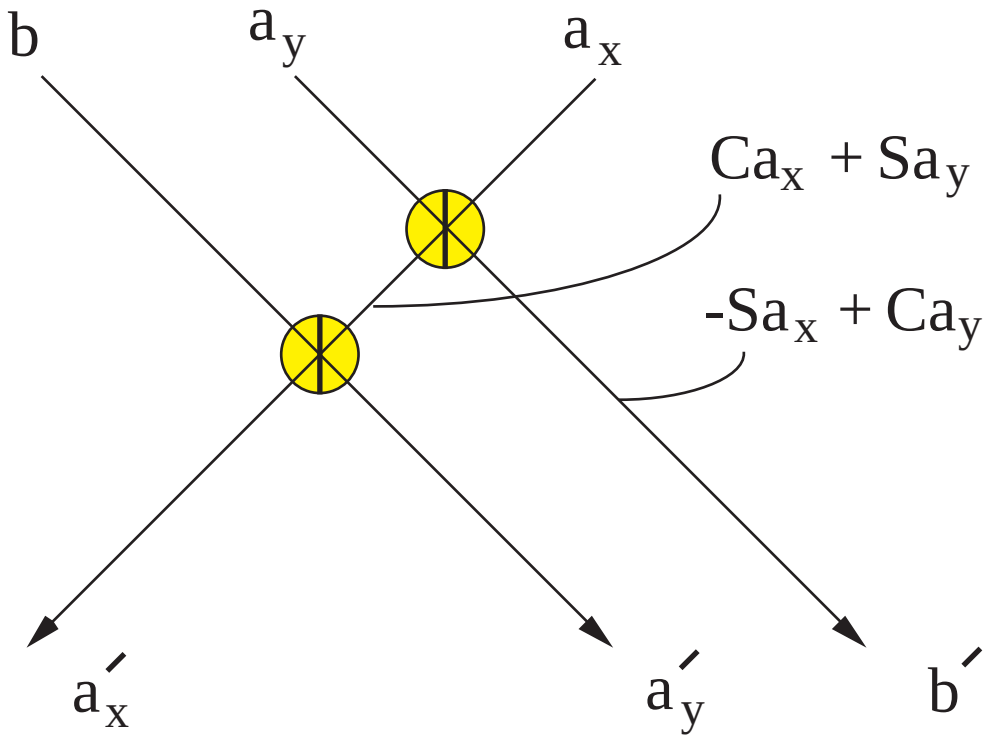

FIG.2

The number of photons in the b mode is assumed to be zero. Thus, only darkness (vacuum) enters the b port of Fig.2. Let M_x, M_y, N_x N_y and N_{loss} be the number of photons in the a_x, a_y, a'_x, a'_y and b' modes, respectively. Then the amplitude Amp_{pol} of the polarizer is

$$\mathrm{Amp}_{pol} = \langle 0| \frac{(a'_x)^{N_x}}{\sqrt{N_x!}} \frac{(a'_y)^{N_y}}{\sqrt{N_y!}} \frac{(b')^{N_{loss}}}{\sqrt{N_{loss}!}} \frac{(a_x^\dagger)^{M_x}}{\sqrt{M_x!}} \frac{(a_y^\dagger)^{M_y}}{\sqrt{M_y!}} |0\rangle \ . \tag{9}$$

If in Eq.(9) we express the primed modes in terms of the unprimed ones, we find that:

$$\mathrm{Amp}_{pol} = C^{N_x} S^{N_y} \sqrt{\frac{(N_x+N_y)!}{N_x! N_y!}} \mathrm{Amp}_{bs} \ , \tag{10a}$$

where

$$\mathrm{Amp}_{bs} = \langle 0| \frac{(Ca_x + Sa_y)^{N_x+N_y}}{\sqrt{(N_x+N_y)!}} \frac{(-Sa_x + Ca_y)^{N_{loss}}}{\sqrt{N_{loss}!}} \frac{(a_x^\dagger)^{M_x}}{\sqrt{M_x!}} \frac{(a_y^\dagger)^{M_y}}{\sqrt{M_y!}} |0\rangle \ . \tag{10b}$$

Amp_{bs} is a beam-splitter amplitude. It can be evaluated by expressing it as a series. See the appendix entitled "Beam-Splitter" for the details. Quantum Fog uses Eqs.(10) to calculate Amp_{pol}.

The values of Amp_{pol} for one incoming photon are as follows:

One Photon Amplitudes :

	$M_x = 1, M_y = 0$	$M_x = 0, M_y = 1$
$N_x = 1, N_y = 0, N_{loss} = 0$	C^2	CS
$N_x = 0, N_y = 1, N_{loss} = 0$	CS	S^2
$N_x = 0, N_y = 0, N_{loss} = 1$	$-S$	C

The one photon amplitudes may be represented graphically as follows:

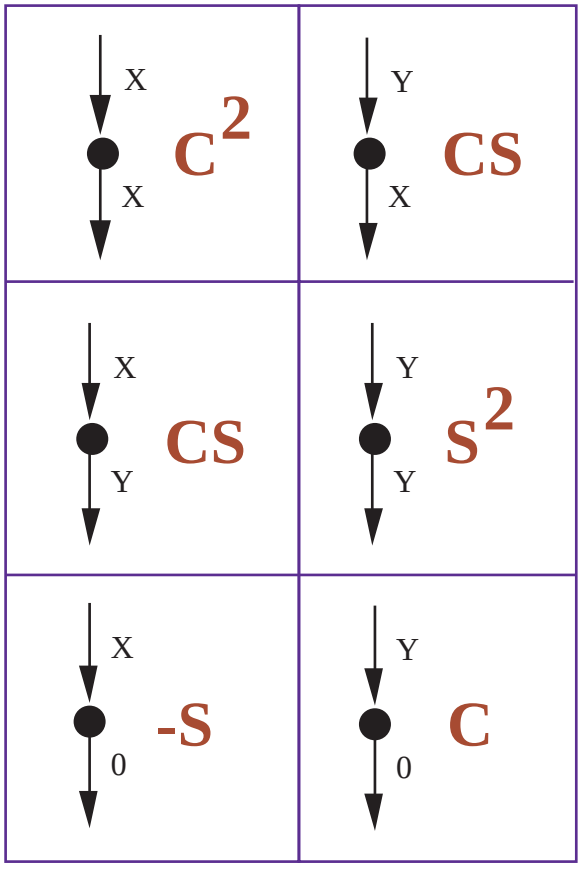

FIG.3

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Qbit Rotator

Suppose $M \in \{0,1\}$ is the state of the single qbit entering a qbit rotator node, and $N \in \{0,1\}$ is the state of the node itself. Suppose $\theta_0, \theta_x, \theta_y, \theta_z$ are real numbers and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Then the amplitude Amp of the qbit rotator is given by the (N, M) entry of an exponential matrix:

$$\text{Amp} = \left[e^{i(\theta_0 + \vec{\theta}\cdot\vec{\sigma})} \right]_{N,M} \ . \tag{1}$$

The operator on the right side of Eq.(1) is equivalent to: (a phase shift by θ_0) times (a rotation in the direction $\hat{\theta}$ by an angle 2θ, where $\theta = |\vec{\theta}|$ and $\hat{\theta} = \frac{\vec{\theta}}{\theta}$):

$$e^{i(\theta_0 + \vec{\theta}\cdot\vec{\sigma})} = e^{i\theta_0} R_{\hat{\theta}}(2\theta) \ . \tag{2}$$

The operator on the right side of Eq.(1) can also be expressed as

$$\begin{aligned} e^{i(\theta_0 + \vec{\theta}\cdot\vec{\sigma})} &= e^{i\theta_0}(\cos\theta + i\hat{\theta}\cdot\vec{\sigma}\sin\theta) \\ &= e^{i\theta_0} \begin{bmatrix} C + i\frac{\theta_z}{\theta} S & iz^* S \\ izS & C - i\frac{\theta_z}{\theta} S \end{bmatrix} \end{aligned} \quad , \tag{3}$$

where

$$\theta = |\vec{\theta}| \ , \tag{4a}$$

$$C = \cos\theta \ , \quad S = \sin\theta \ , \tag{4b}$$

$$z = \frac{\theta_x + i\theta_y}{\theta} \ . \tag{4c}$$

Quantum Fog uses Eqs.(3) and (4) to calculate the amplitudes of a qbit rotator.

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Stern-Gerlach Magnet

The appendix entitled "Spin 1/2 Particles" gives a formula for evaluating $\langle \sigma_{u'} | \sigma_u \rangle$, where $\sigma, \sigma' \in \{+, -\}$, and where $\hat{u}'$ and $\hat{u}$ are any two unit vectors. Quantum Fog calculates the amplitude of an S.G. magnet node in terms of the formula for $\langle \sigma_{u'} | \sigma_u \rangle$. It does this as follows.

We define type 1 and type 2 parent nodes as follows:

- type 1: A node that can assume two states named "0" and "1". Here 0 and 1 refer to the number of particles (spin 1/2 fermions) in the state.
- type 2: A node that can assume three states named "(0,0)", "(0,1)", and "(1,0)". The first component refers to the number of spin $-$ particles in the state, and the second to the number of spin $+$ particles in the state. For example, (1,0) represents a state with one spin $-$ particle.

(Mnemonic: type 1 has 1 component, type 2 has 2 components).

Consider an S.G. magnet node. Let $X_1, X_2, \ldots$ represent its parent nodes. The nodes X_j must be either of type 1 or type 2. Define $X_\cdot$ by $X_\cdot = (X_1, X_2, \ldots)$. Let N_{in} represent the number of particles entering the magnet node. In case that $N_{in} = 1$, let $\hat{u}$ represent the quantization direction of the single input particle, and let $S_u \in \{+, -\}$ represent the projection along $\hat{u}$ of the particle's spin. Let $\hat{b}$ represent the direction of the magnetic field of the S.G. magnet. Let (N_{b-}, N_{b+}) represent the state of the magnet node. N_{b-} (ditto, N_{b+}) is the number of particles exiting the node with spin projection $-$ (ditto, $+$) along $\hat{b}$. Then Quantum Fog evaluates the amplitude $A[(N_{b-}, N_{b+})|X_\cdot]$ of the magnet node according to the following table:

$\mathbf{A}[(\mathbf{N_{b-}}, \mathbf{N_{b+}})|\mathbf{X}_\cdot]$

	$N_{in} = 0$	$N_{in} = 1$	$N_{in} > 1$
$N_{b-} = 0, N_{b+} = 0$	1	0	0
$N_{b-} = 1, N_{b+} = 0$	0	$\langle -_b \vert S_u \rangle$	0
$N_{b-} = 0, N_{b+} = 1$	0	$\langle +_b \vert S_u \rangle$	0

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